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Color Work to Enhance Proof-Writing in Geometry

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1. Introduction

Classroom teachers know that the use of color can be a powerful tool to keep track of and make sense of mathematical information. The introduction of colored chalk as well as overhead transparencies and colored markers into instruction in the second half of the 20th century was a revolution in teaching tools. The current uses of whiteboard and SmartBoard® with colored markers continues this tradition in instructional tools. The use of manipulatives and, to some extent diagrams, among learners has been researched and incorporated into the toolbox offered to current and future teachers (e.g., Friel & Markworth, 2009; Smith, Hillen, & Catania, 2007). However, the potential benefits of the uses of color in mathematics learning have not been systematically researched. While the research is new, the idea is not.

In 1847, Oliver Byrne published his reworking of Euclid's Elements, in which he used colored diagrams so extensively that the visual representations were inseparable from the proofs they were intended to support (see Figure 1). Published at a time when geometers' attention focused on non-Euclidean investigations, Byrne's work was not taken seriously, and was "regarded as a curiosity" (Cajori, 1928, p. 429). Byrne, however, did not intend his work for mere entertainment, but said the book enhanced pedagogy and encouraged retention of mathematical ideas by appealing to the visual. He suggested that by communicating Euclid's ideas through colorful renderings, instruction time could be used more efficiently and student retention increased (Byrne, 1847). One hundred and fifty years later, *why* this is the case is finally coming to be understood.

There are many ways that the use of color can reduce the difficulty of a problem situation without decreasing its cognitive complexity. Just as thinking about a phone number as three chunks of numbers allows us to remember a long string of

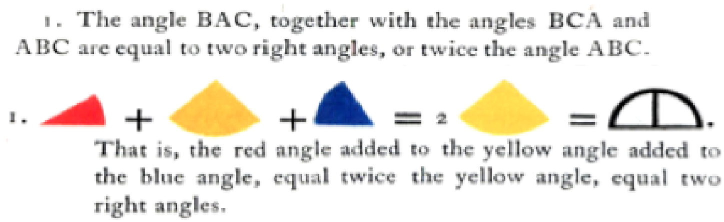


FIGURE 1. Example of a geometry statement first shown in words (translated from how Euclid expressed it), and then in colored shapes as Byrne (1847) represented it.

digits, the use of color can simplify the load on working memory and allow a learner to represent and strategize more efficiently (Paschler et al., 2007). Recent work in the learning sciences suggests that carefully selecting color in visual representations and combining information in a figure or symbolic expression can promote the integration of concepts. When presented with multiple sources of information (e.g., when a teacher relates parts of a mathematical equation to a graph or a student interprets a diagram), learners must direct their attention to each individual source, encode separate pieces of information, and then manage the stored information to make meaningful connections. Splitting attentional resources is cognitively demanding and may serve as an obstacle to learning. In fact, clinical research on the use of diagrams indicates that when individual sources of information are visually integrated, student learning is improved (e.g., Bobis, Sweller, & Cooper, 1993).

The work to date on color-coding for understanding symbolic grouping is further along than the work on color use in figures. In their research with pre-service elementary teachers, McGowan and Davis (2001) observed that students initially struggled to move from concrete manipulatives to algebraic expressions, and also struggled to see connections to binomial expansions. One student, however, conjectured that binomial expansions such as

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

could be re-interpreted through the use of two colors. The student represented a particular problem with a black and white color scheme and substituted black for a and white for b and restated the second-order equation using B for black and W for white, as $(B+W)^2 = B^2 + 2BW + W^2$. This idea resonated with the rest of the class and appeared in their subsequent work, indicating to the authors the algebraic symbols had “become genuinely symbolic – symbolizing something” (p. 441).

In working with secondary students to find a general rule for sequences of numbers, Waring (2008) used color to highlight relationships in pictorial representations of the sequences. Using red and blue to differentiate squares within each figure, students were able to correctly identify a sum of the squares of two numbers that related back to the figure number, n (see Figure 2).

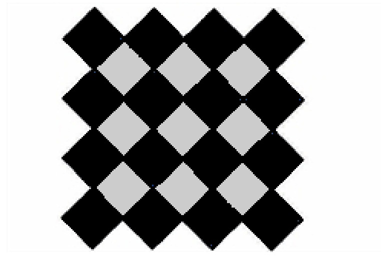


FIGURE 2. Red and blue (here, grey and black) coloring in the 4th figure of a sequence enabled students to see that the nth figure generalized to $n^2 + (n-1)^2$.

In research on student use of monochromatic figures, Gibson (1998) found that students use diagrams in several complementary ways:

- to understand information,
- to determine the truthfulness of a statement,
- to discover new ideas, and
- to verbalize their thinking.

Yestness (2012), in extending Gibson's work, noted that undergraduate students felt that their drawings were for personal use and not for proof or explanation. Nonetheless, when asked to explain a proof, students (and mathematicians) will draw one or more diagrams to support an explanation (e.g., Burton, 2004; Samkoff, Lai, & Weber, 2012). In fact, compact figural representations appear to be an intuitively powerful component of mathematical learning in the context of proofs and proving.

2. Nuances of Representation (models of) and Strategy (models for)

In investigating students' routes from informal mathematical activity to formal mathematical reasoning, Zandieh and Rasmussen (2010) explore models as "student-generated ways of organizing their activity with mental or physical tools" (p. 74). In particular, they specify a difference between models-of mathematical activity and models-for mathematical reasoning. It may be that students can use color in constructing diagrams of geometry proofs in this way – as a tool of representation, as well as a strategic tool for understanding.

In fact, the *Common Core State Standards* for mathematics, particularly the Mathematical Practices, highlight the kind of thinking supported by intentional color-coding. As noted in the Geometry strand, high school geometry students build upon elementary and middle school geometry content as they construct mental models for precise definitions and develop strategies for generating and validating proofs (CCSSO, 2010). In particular, the Mathematical Practices indicate that high school students are expected to: develop skill in abstract and quantitative reasoning, which includes practice creating representations of problems (Practice 2); construct arguments and evaluate others' arguments, which includes understanding and employing definitions, assumptions, and previous results for constructing arguments, while also communicating about and evaluating others' results (Practice 3), and appropriately and strategically using a variety of tools, such as paper and pencil (colored or standard), ruler, protractor, and dynamic geometry software (Practice 5).

3. Illustrating the Ideas: The Case of Charlotte

Here, we share what we are learning in our research on coloring and proofs. In particular, we focus attention on Charlotte Knight (a pseudonym), an undergraduate mathematics major preparing to be a secondary mathematics teacher, and her work while enrolled in a college course focused on modern geometry. Charlotte regularly employed coloring techniques in her proof-writing that were similar to the proofs offered by Byrne. Charlotte's representations enhanced her understanding in a way that may be of value to K-12 teachers and their students. We met with Charlotte for a task-based interview with two main components: first a review of one of the original colored proofs she submitted, in which she correctly proved that the diagonals of a parallelogram bisect each other, and then work on a proof covered in class, the Pointwise Characterization of Angle Bisectors Theorem:

> *Let A, B, and C be three non collinear points and let P be a point in the interior of $\angle BAC$. Then P lies on the angle bisector of $\angle BAC$ if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$.*

We had colored pens available on the table for her to use. Charlotte spent about 30 minutes of her 75-minute long interview describing how and why she used color to enhance her proofs. She also used color extensively in generating her proof of the Angle Bisectors Theorem (about 25 minutes). All four aspects of diagramming offered by Gibson (1998) and supported by Yestness (2012) were apparent in Charlotte's colored proofs.

In particular, Charlotte relied most on color in determining the truthfulness of statements and writing out ideas. She used color to confirm or refute ideas and to document the pathways she took. She also used it to reduce her cognitive load – she found it less mentally taxing to use color (rather than symbols or words). Including color served to help her sort and organize relationships, which she then used to write out her proofs. Charlotte used colors in two ways:

(1) as an organizational tool to connect her diagrams to the content of her proofs (i.e., as a tool of representation) and
(2) as a reasoning tool to understand the theorem (i.e., as a tool for understanding).

3.1. Color as a tool of representation. In the proof she was asked to recount, where she proved the diagonals of a parallelogram bisect each other, Charlotte employed a 4-color scheme. She used these colors in a way in which the diagram was inseparable from the proof it was intended to accompany; she colored the angles to correspond to the underlined colors in her proof (see Figure 3).

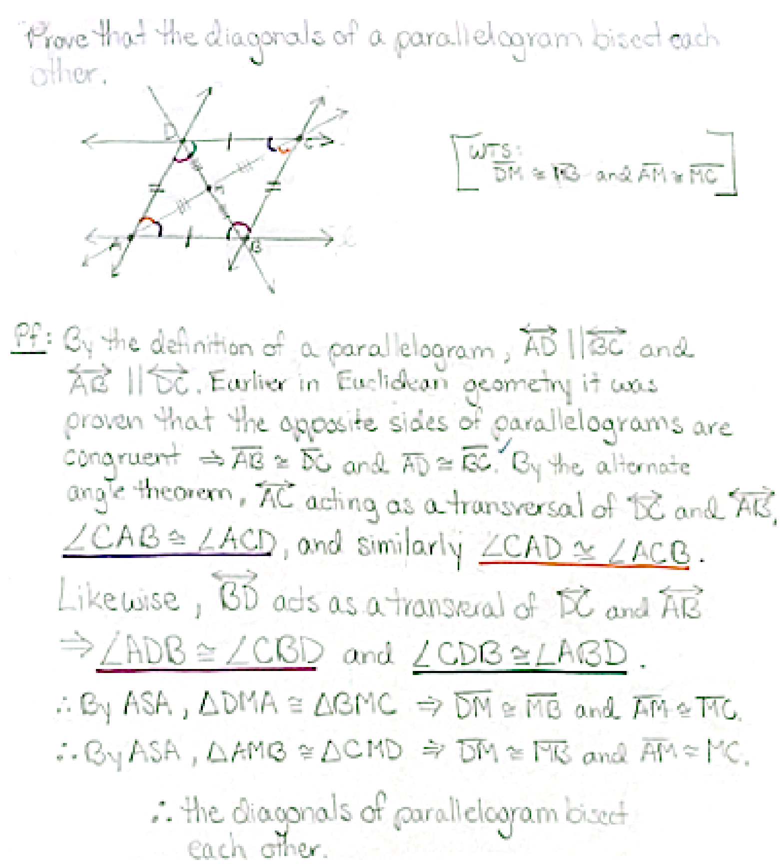


Figure 3. Charlotte's colored proof of the statement that the diagonals of a parallelogram bisect each other.

In describing this proof, she used the language "purple is congruent to purple," "orange is congruent to orange," "pink is congruent to pink," and "green is congruent to green." That is, the color replaced the alphabetical identifiers and this is how Charlotte navigated her proof:

> *I needed to look at, like, labeled the purple angles and then I underlined them for both so I knew purple was done . . . now which one is similar to the purple ones . . . to the orange ones and then I have pink and green left, well pink and then which one is similar to pink? Green. So that's how that, that's how that went.*

In recounting this proof, she spoke primarily of using colors to organize her ideas and understand the information required to write the proof.

3.2. Color as a tool for understanding. Charlotte regularly used color to indicate "direction" in a theorem. She did this when proving "if and only if" theorems, saying she was uncomfortable with these because she had difficulty keeping track of which "direction" she was proving, what information she could assume, and what she was trying to show. In her proof of the Pointwise Characterization of Angle Bisectors, Charlotte used a 2-color scheme. All information in the necessary "direction" was designated green and all information in the sufficient "direction" was designated blue. She then constructed and colored a diagram to reflect this information. As a result, to Charlotte, the statement of the theorem changed from "Then P lies on the angle bisector of $\angle BAC$ if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$" to "Then green if and only if blue" (see Figure 4). This served to help her reduce the cognitive load of attending to both implications in the "2 direction" theorem. It also aided her understanding of the information required to construct a proof:

> *I'm not as familiar with this picture . . . so I needed to keep referencing back and forth here and so I needed to know . . . it's kind of like a help to know where I'm going and it's, it's a reference.*

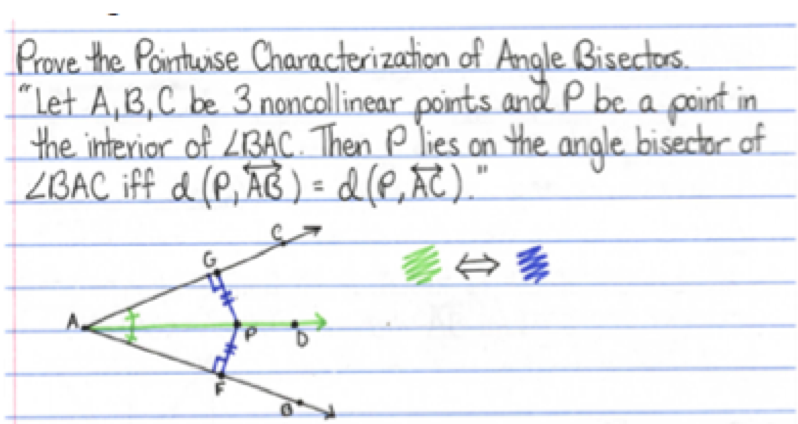


FIGURE 4. Charlotte's reworking of the statement of the Pointwise Characterization of Angle Bisectors theorem to "green if and only if blue" in a class homework assignment.

Charlotte said using the color helped her stay organized, understand the theorem, and stay on track with her proving goals:

> *This helps me remember which direction I'm going, 'cause all the green stuff is what I knew from the first half of the statement . . . I put all of that in green.*

As she continued to construct the proof, Charlotte added a second layer of coloring – one in which she used color to understand and manage the mathematical content (see Figure 5).

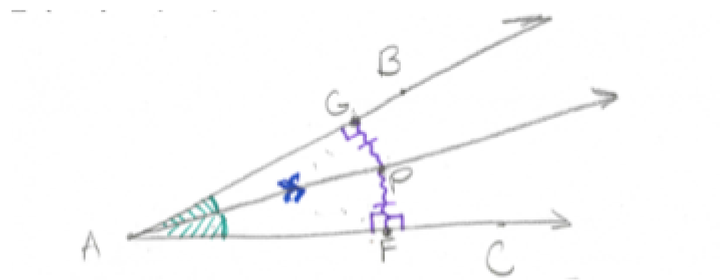


FIGURE 5. Charlotte's second drawing for the Pointwise Characterization of Angle Bisectors Theorem for "green is congruent to green if and only if purple is congruent to purple."

> *I don't have [segment] AG, I don't know anything about [segment] AG so there's no colors or label, there's no – nothing. I don't know anything about [segment] FA. What I do know is all in color, so it kind of helps me know, well, this is what I have to work with, because I don't want to go try to prove [segment] FA and [segment] FG, I don't have anything to work with to get there, so it helps that I have the purple angles here to say these are right . . . I don't think I used anything that wasn't related to color in some way. Like I'd never talked about just the segment FA, you know what I'm saying? I talked about segment AP, but I gave it a blue squiggle.*

4. Discussion

In advanced mathematics the prevailing wisdom is that pictures cannot prove. Students are discouraged from relying much on their visualizations when it comes to proofs and proving (Brown, 1997; Hanna, 2000). Charlotte agreed with this sentiment. She felt it was valuable to have a colored proof for her own sense-making. That is, a statement such as "If blue is congruent to blue, and purple is congruent to purple, then red is congruent to red" might be good for her notes. However, she asserted that without shared meaning, a proof such as this would not be a correct proof for "mixed company." Not only does Charlotte's view echo Byrne, it also illustrates something Martin Gardner said several years ago, "There is no more effective aid in understanding certain algebraic identities than a good diagram. One should, of course, know how to manipulate algebraic symbols to obtain proofs, but in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance" (Gardner, 1973).

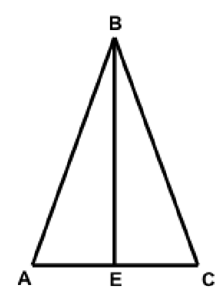
Mathematicians have the mathematical language mastery that allows them to navigate the formal symbolism of proofs. For students, use of the kinds of color-coding in visual representations discussed here may enhance understanding and may even serve as a viable proof-prepratation tool (Arcavi, 2003). While Byrne's (1847) assertion that using color-coding would allow students to see, at a glance, key parts of an argument generally has been affirmed by 20th century research on the mental "chunking" we do to manage complex information, Charlotte's work provides substantial support for this in the context of Geometry. Additionally, we noticed a growing number of students employing the use of color to support their diagrams in our advanced undergraduate mathematics classes – particularly those in which a majority of the students enrolled were seeking secondary mathematics teaching licensure.

As noted in the *Common Core State Standards*, some students use their experiences in high school geometry to develop Euclidean and non-Euclidean geometries as axiomatic systems. When students go on to college and prepare to become teachers, a collegiate geometry course is where students gain essential skills in visualization for "understanding the nature of axiomatic reasoning" and "facility with proof" (CBMS, 2000, p. 41). Yestness (2012) has observed that expanding pre-service teachers' experiences to include color-coding as a tool for their own learning, may "expand their pedagogical choices as teachers" (pp. 226-227).

5. Recommendations for Implementing Color in the Classroom

Although Charlotte was enrolled in an undergraduate modern college geometry course, high school geometry provides a similar context for teaching with color-coded proofs. The techniques might also be modified for use in the middle school classroom to prepare students for the transition to writing proofs in high school. Through our experiences using color to inform proof-writing in geometry courses at the undergraduate and secondary levels, we have identified five essential components for implementing color. We illustrate these recommendations by way of an example, generating a proof for the following:

Given:
Triangle ΔABC is an isosceles triangle with base $\overline{AC}$.
Segment $\overline{BE}$ is a median.
Prove:
Segment $\overline{BE}$ is an angle bisector.



5.1. Communication. When incorporating color as a tool for understanding, explicitly identify and communicate the strategy. Multiple strategies may emerge during the proving process. Be specific about the use of color. For example, in the figure below, we employ color as a tool for understanding the prompt. All given information is colored green in the diagram and indicators for the statement that is to be proved are drawn blue, thereby changing the prompt to "If green is true, then blue must be true" (see Figure 6).

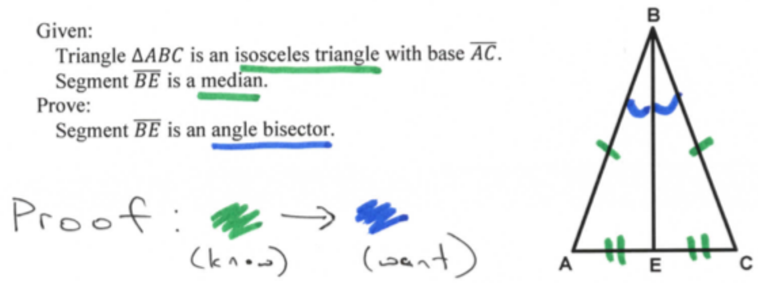


FIGURE 6. The statements in the prompt translate to "If green, then blue" in the diagram.

Continuing this process means the coloring scheme expands. It includes more colors as we use the scheme as a tool for representation (see Figure 7).

5.2. Purpose. Every color that you use should have a purpose – it captures some shared characteristics of the labeled parts. For example, in the diagram below, segment AE is congruent to segment EC. The purple double-ticks on AE and AC in the figure show congruency. We do not use purple again because there are no other segments necessarily congruent to these. We use green, red, pink, and blue in similar ways (see Figure 7). The monochromatic use of single or double ticks is enhanced with color as a tool for reasoning.

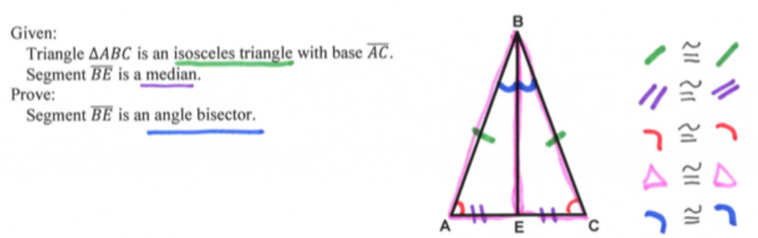


FIGURE 7. The diagram is colored to show "green is congruent to green" and "purple is congruent to purple." Therefore "red is congruent to red" and "pink is congruent to pink." Thus "blue is congruent to blue," completing the proof.

5.3. Incorporation. After completing a color-coded analysis of a figure or set of figures, the subsequent written proof should also incorporate the colors used in the figure analysis – either by writing, underlining, or highlighting in the appropriate colors. For example, in the two-column proof in the figure below the two congruent angles $\angle BAC$ and $\angle BCA$ are colored red in the diagrammatic proof. This is noted in the corresponding two-column proof by underlining the congruence statement in red. Other statements are similarly underlined in green, purple, pink, and blue (see Figure 8).

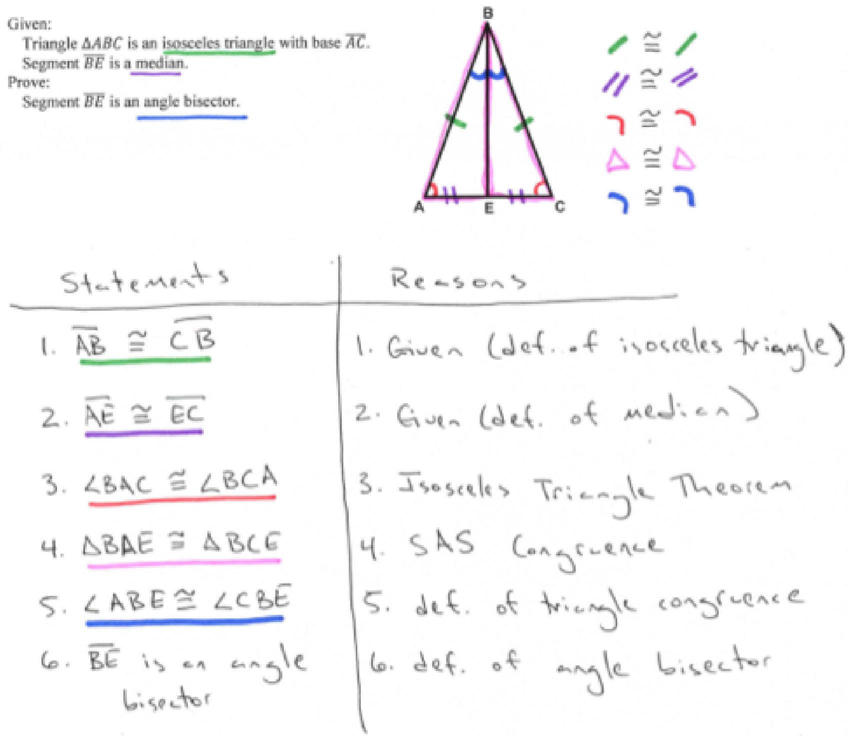


FIGURE 8. The "colored proof" has been translated into a colored two-column proof.

5.4. Consistency. When using color to teach geometry, be consistent with color use. Always make a legend to label the use of color, and consistently explain the property or characteristic captured or represented by the color in that color and in words. Students will be more inclined to use color in their learning process when it is a consistently modeled for understanding and representation. In Figure 6, in addition to coloring the "given" statements green and the "prove" statement blue, we included a note to accompany the diagram. In Figure 7, we included a colored legend to indicate congruences. This assisted us when we translated our colored proof into a traditional two-column proof in Figure 8.

5.5. Resources. Provide color tools in the classroom such as colored pencils or markers. This provides an equal opportunity for all students to participate by using color. Note that students may be more inclined to use color in their personal work when the tools are reliably available and their use expected in the classroom.

6. Conclusion

The utilization of color is not intended as a method for getting students' attention or as a means to make complicated drawings more attractive. Rather, through color-coding, relevant information in a proof is highlighted and significant relationships among components are foregrounded. Proof coloring is also beneficial for students to use as a learning tool on their own. A student can choose, strategically, how to color accompanying diagrams.

The strategies we have illustrated here are to color-code as a tool (1) of representation for facts and (2) for understanding of relationships. Such color-coding can assist students in packing and unpacking information and managing the complexity of proofs and proving. Furthermore, it may be that teachers can better assess how a student is approaching proof writing based on the color scheme utilized by the student. It provides teachers with a tool to help communicate with individual students and their different approaches to proof writing.

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