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A Pedagogical Framework for Distinguishing Mathematical Definitions in the Classroom

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Abstract. Most students believe that definitions play an important role in the learning of mathematics. However, they struggle with transitioning from performing procedures to reasoning from definitions as they move into more sophisticated mathematics. Further, students tend to focus in on their experiences, examples, and intuition, especially the visual/mental images they hold for a concept, instead of the axiomatic assertions provided by a mathematical definition. This disconnect between approaches needs to be addressed at all levels of study. The goal of this paper is to present a pedagogical framework and some examples on how to address the role of definition in the study of mathematics for students at all levels.

Introduction

What is the definition of a circle? My class of pre-service teachers agreed that a circle is the set of all points equidistant from some center point C. Through some class discussion they realized it was important to clarify that this definition is inaccurate if we do not state that it holds on a two-dimensional plane. Immediately after the conversation about the definition of a circle ended I asked the class to give me examples of circles. Cup! Soccer Ball! Polka Dot! Stop Light! Coin! They shouted these out with glee as I noticed they were completely unaware of how their mental image of circle overrode the key points of a definition we just spent five minutes deliberating. How could I help my students recognize that being sloppy with mathematical definitions is not going to bode well for them as teachers or learners of mathematics?

While attending primary school I was taught a variety of definitions in mathematics. However, most of the definitions that were used were more informal in nature (e.g., a triangle is a three sided figure). It was not until I was an undergraduate majoring in mathematics taking a linear algebra class that I discovered the importance of more formal definitions in mathematics.

This discovery did not come from my professor stating explicitly to my class the necessity of learning mathematical definitions or their importance in the learning of linear algebra. Rather it was born out of the fact that I was getting a C in the class and really had no idea why I was performing so poorly. I sought out the help of my professor and together we discovered that I completely ignored the definitions at my disposal. In fact, I was not even clear on how they would help me to do my homework. It was at that point that my professor clarified to me not only the importance of knowing what the definition states but what information it tells me about the concept itself. It was made clear that understanding the formal mathematical definitions would be critical in not only helping me to solve my homework problems in his class but were also necessary for me to be successful in all of the upper division mathematics courses I would be taking due to the proof writing aspect of the courses.

This was a huge *Aha!* moment for me in my development as a learner of mathematics. And I believe that it has been a bit of a rite of passage for many scholars in mathematics. However, I do not believe that professors of mathematics knowingly keep this information from their students. In fact, the use and importance of mathematical definitions is so deeply engrained in their culture that it is likely assumed their students already know the role and use of mathematical definitions. This is true even though professors of mathematics also recognize that their students do not "know" mathematical definitions (Alcock & Simpson, 2002; Edward & Ward, 2008). I would also posit that because the role and use of mathematical definitions is not made clear to students, it could be a reason why any first course in mathematical proof (for me it was linear algebra) tends to be the course that weeds out mathematics majors from completing a degree in mathematics.

Background

The creation and use of mathematical definitions is quite different from our "everyday language" definitions. Edward and Ward (2004) distinguished two different types of definitions: extracted (everyday language) and stipulated (mathematical) definitions. *Extracted definitions* are inductive, "based on examples of actual usage, definitions extracted from a body of evidence" (Landau, 2001, p. 165). Whereas *stipulated definitions* are axiomatic assertions, "setting up of the meaning-relation between some word and some object, the act of assigning an object to a name (or a name to an object)" (Robinson, 1962, p. 59).

In their study, Edward and Ward (2004) found that undergraduate mathematics majors often do not view a mathematical definition as stipulated like a mathematician would. Moreover, it was discovered that students would choose an extracted definition approach, and their intuition about a concept, over a given, stipulated definition, when reasoning about mathematical tasks. That is, the participants in the study stated and explained the stipulated definitions needed to perform a mathematical task, but were unsuccessful in their attempts to complete the task. Knowing how to state definitions is not enough to be successful in performing mathematical tasks, especially those involving proof writing.

Pre-service Elementary Teachers

My experience training prospective elementary school teachers has revealed similar ways of thinking about definitions in mathematics. While pre-service teachers believe that definitions play an integral role in the learning and teaching of mathematics, they notoriously ignore the constraints in a stipulated definition. When reasoning, they rely primarily on their experiences, examples, and intuition, especially the visual/mental image they hold for the concept. For example, while teaching a Geometry and Measurement course for pre-service elementary teachers I am regularly surprised by scenarios like the one at the start of this paper. None of the examples of circles. Rather, they are examples of items that have a circular quality. This exemplifies how the extracted approach will often take precedence over the stipulated, rigorous mathematical definition when working on mathematical tasks.

In a study among pre-service teachers, Ward (2004) posited that the future teachers in her study needed a mathematical "intervention." Specifically Ward recommended that students should see examples of concepts that are not always traditional in appearance. She suggest that unconventional examples would help to develop an approach to noticing and using definition that is more consistent with standard mathematical use. Further, Ward suggested that teacher educators teaching mathematics content courses are on the front lines, facilitating the breaking of the cycle. I would go one step further and suggest that the disconnect between approaches to definitions needs to be addressed at all levels of study in elementary, secondary, and post-secondary mathematics. The goal of this paper is to present a framework, with some examples about key aspects of definitions, and offer an activity to address the role of definition in the study of mathematics for students at all levels.

Pedagogical Framework

The role of definition must be explicitly discussed with students. Caution is needed in determining what students understand about the role of definition in mathematics. Many students believe that without at least a basic understanding of a definition it can be hard to solve problems involving those definitions. However, few have experience with understanding and working from mathematical definitions as a major aspect to being successful in mathematics. Therefore, educators must create opportunities for students to see the importance of mathematical definition not only in communicating mathematical ideas but, more importantly, in reasoning from those definitions to construct conjectures, provide sound mathematical arguments, make connections between concepts, and to discover new mathematics (Cuoco, Goldenberg, & Mark, 1996; Leikin & Zazkis, 2010).

Repeated exposure to the importance of mathematical definition is necessary. I was once told that a belief is just a thought that you thought a lot. Educators can play a central role in helping to shape student beliefs about the role of definition in mathematics by implementing frequent opportunities for students to explore mathematical definitions. In fact, this can be a thread woven throughout a course. The sooner and more times students are exposed to mathematical definitions as a mathematical habit of mind, the better.

Distinguishing a mathematical definition is an essential mathematical habit of mind. Exploring mathematical definitions can have many facets. One method that I focus on here, and that I have found the most powerful in my classrooms, is to distinguish the definition. Distinguishing a definition is quite simply determining what a mathematical concept is and what it is not, identifying examples and non-examples of it. Distinguishing a definition provides an opportunity to tease apart the necessary and sufficient conditions of a mathematical concept so students will have an understanding of the edges and constraints of a defined concept.

What is more, distinguishing a definition can assist the learner in turning abstract ideas into working knowledge. This is accomplished by focusing student attention on each of the necessary and sufficient components of a definition and exploring how augmenting or removing any one of the conditions alters the concept in a nontrivial way (Cuoco et al., 1996). This process plays a significant role when determining when objects are members of a category. This is particularly valuable when reasoning from definitions in a proof writing context.

Providing non-traditional examples and non-examples of concepts helps students to gain a richer and more nuanced view of the concept. Non-traditional examples include extreme cases of the concept or non-traditional orientations of the concept (e.g., non-gravity based and/or irregular shaped polygons).

For example, when distinguishing box and whisker plots with students instead of providing only a traditional example (see Figure 1), educators can also include several non-traditional examples that will foster a broader view of a box and whisker plot (see Figure 2, next page).

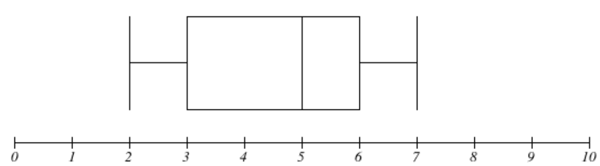

Figure 1. Traditional example of a box and whisker plot.

In Figure 1 we can see that the five number summary has all distinct values. However, in each of the examples in Figure 2 we can see that something peculiar is happening. Most students will not connect that oddity to the five number summary. Students' natural assumption is that the plots in Figure 2 are non-examples. With some questioning about what it might mean about the data if there is a missing whisker, students will eventually come to the conclusion that the third quartile and the maximum in the lefthand plot in Figure 2 must share the same value and therefore the upper 25% of values might all be the same value, namely 5. In the righthand plot in Figure 2, students begin to realize that not only are the

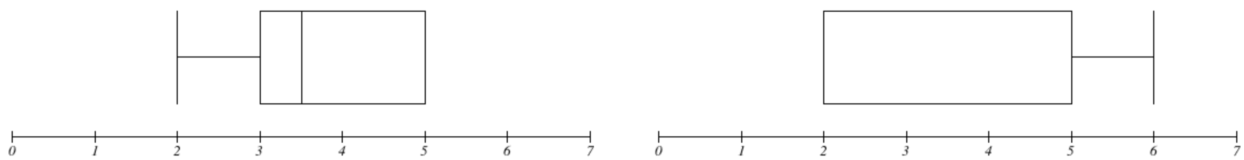

Figure 2. Non-traditional examples of box and whisker plots.

first quartile and the minimum equal but the median must also share that same value. Further, the lower half of the data must consist of all the same values, in this case 2.

Be intentional with definitions in the classroom. Illustrating the misuse of a definition can provide a compelling argument for being intentional with our word choices in a mathematics course. In the classroom, this might look like choosing our words carefully and intentionally to create cognitive dissonance for our students. For example, when I introduce measures of central tendency I purposely provide my students with a classic example of using the word "average" to confuse and mislead. Most people take average to be the same as the arithmetic mean. However, through a whole class discussion students come to understand that the word average can refer to mean, median, or mode. This conversation provides an opportunity for students to see how important it is to say what they mean and mean what they say in a mathematics classroom. Intentionally providing opportunities for students to see a misuse of a definition can help them to come to understand rather quickly that they need to be precise with their word choices in a mathematics classroom when referring to mathematical concepts.

This four pronged pedagogical framework of attending to (a) the role of definition, (b) repeated exposure to the use of definitions as problem-solving and reasoning tool, (c) distinguishing a definition, and (d) intentional use of language in working with definitions in challenging ways, requires educators to make mathematical definition and its importance to the learning and understanding of mathematics a pervasive theme in the classroom. Further, the framework promotes active and deep engagement with definitions to encourage reasoning and sense making from definitions.

Coochy-Coo Activity

To provide a meaningful experience regarding the role of definition in mathematics for a workshop with in-service elementary teachers, I developed the Coochy-Coo activity. The activity places students in the unique position of learning mathematics using a new language in which they must be able to communicate and reason. This activity brings students through three different levels of understanding of a mathematical definition. The first level is the informal definition, the way a student might think of the concept focusing on the most basic aspects of the definition. The second level is the typical school or textbook definition. This level is slightly more rigorous than the informal definition. However, it may lack some of the necessary and sufficient conditions that the third level of definition would entail, the formal mathematical definition, that is used to reason about mathematical concepts.

In the Coochy-Coo activity, students are working to understand what a coochy-coo (i.e., a heptagon) is in a new language. They are given a student definition that purposely does not use the word polygon and are asked to determine from a group of figures which ones are coochy-coos based on this informal definition. Next students are given a textbook definition that is filled with even more words with which students are not familiar. Students are then provided with additional textbook definitions in English to help clarify what they meanings of terms are. For example, students are told that "A coochy-coo is a tiddly-wink enclosed by seven biggity-bee chack-taks." Students are then provided with definitions of tiddly-wink (a region) , biggity-bee (straight), and chack-taks (line segments). They must piece these together to create a working definition of a coochy-coo so they can further distinguish it from a group of figures using this new definition.

Lastly, students are provided a formal mathematical definition completely in English, though it may contain words that are new to (e.g., collinear or closed polygonal path). The formal definition has all the necessary and sufficient conditions to identify a figure as being a coochy-coo. Students then reexamine all the figures to identify which are coochy-coos using the formal mathematical definition. The purpose of requiring students to go through all three levels of definition is to assist them in recognizing that mathematical definitions build on each other and one must understand all of the mathematical terms in a definition to truly understand the new concepts being defined.

The last task for students is to come up with their own examples and non-examples of coochy-coos. The purpose of this part of the activity is to further facilitate a student's ability to distinguish definitions through developing unique nontrivial examples and non-examples. The activity further encourages students to start looking at non-traditional examples of definitions. This fosters an awareness that will be more consistent with the formal mathematical definition.

The Coochy-Coo Activity has been successfully implemented with in-service and pre-service elementary teachers. Both groups struggled with the formal definition of a coochy-coo (heptagon). Their struggles centered on whether shape E would be considered a coochy-coo or not (see Figure 3).

E.

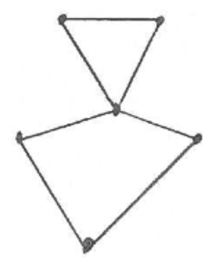

Figure 3. Shape E: Is this a coochy-coo?

Their intuition said no, but it was challenging to figure out how to use the definition to eliminate it. Specifically, they were not reasoning from the definition of a closed polygonal path which states that all points (vertices) must be different (i.e., unique). The vertex that joins the triangular figure to the quadrilateral figure is the issue in this particular shape. Notice that there are four line segments that utilize that one vertex thus the shape is not a closed polygonal path and, therefore, it follows that the figure is not a coochy-coo.

Another troublesome area for some students was coming up with unique non-examples of coochy-coos. Students would initially create figures that were closed or did not have seven sides – making their examples trivial and less interesting. This provided a great opportunity to discuss why it would be important to develop examples that have all the necessary and sufficient conditions of the definition but one subtle piece is missing to help fully distinguish the definition. In our class discussion, I further connected this idea to how new mathematics can be created by examining something familiar but removing a condition that forces it to operate in a new way.

I have noticed my students' relationship to and perception of mathematical definition has changed dramatically. Certainly, it is at the forefront of their minds in my courses. For example, in my geometry course for pre-service teachers, I now find more students refer to specific components of a mathematical definition to support their reasoning about a problem situation. In past semesters, students solely relied on their loosely connected experiences and intuition in an extracted definition approach to reasoning through similar scenarios. In fact, many students have commented on how surprised they were that none of their other mathematics teachers had pointed out how important definitions were in mathematics. The majority of my students stated on midterm evaluations that a focus on definition as a constant component of the work made them more aware of how powerful the definitions were in helping them to be successful in the course.

Implementing the Coochy-Coo Activity early in the semester (or early in a workshop) has helped me to set the level of mathematical rigor required for success in the course while also highlight the importance of definition in the learning and understanding of the material. Mathematical definition has become a theme woven into my courses, strengthening my students understanding of concepts.

Implications for Practice

Employing this pedagogical framework will require teachers to re-examine the curriculum to identify prime locations for implementing activities throughout the content that bring the stipulated mathematical definition to the forefront of learning. Additionally, it means that educators will need to practice refraining from providing the most typical examples of a concept as the first examples. Mathematics educators will be called upon to generate examples in the classroom that highlight the nuances of the mathematical concepts they are responsible for teaching.

The idea of providing more interesting and nuanced examples in the classroom may be challenging for many pre-service and in-service teachers. Therefore, teacher educators will also need to bring attention to stipulated mathematical definitions. Further, they will need to provide opportunities for educators to examine these definitions and to practice developing and identifying thought-provoking examples and non-examples.

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