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Connecting Concepts in Color: Patterns and Algebra

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A mathematical model captures information about the main ideas in a problem situation. The kinds of thinking needed to model with mathematics are central to the Common Core State Standards for Mathematics [CCSSM] (2010). In fact, "model with mathematics" is included in the standards as a valued habit of mind to be cultivated (Standard for Mathematical Practice 4) as well as core content. Modeling includes several kinds of activity: noticing associations among categories and quantities, developing mathematically useful representations of situations, analyzing relationships and representations, taking the risk of starting (and perhaps abandoning) a strategy. Important to modeling are making connections among representations while bringing to mind and choosing strategies for making sense of a scenario (Schifter, Bastable, & Russell, 2009; Seago, Mumme, & Branca, 2004). Clear in the CCSSM and the associated Learning Progression documents is that all of these skills, practices, and habits of mind are sharpened in middle school (Common Core Standards Writing Team, 2015).

We argue that certain uses of color in the mathematical practice of modeling support (a) connections across representations and (b) decision-making among strategies. In particular, students and teachers can use color to keep track of meaning. A common example is writing one equation in red and another in blue and then drawing their graphs in red and blue, respectively. Yet, color can be used for much more.

Our purpose here is pragmatic: to share the evidence-base for using color in certain ways and to describe its use in a particular professional development course. While our examples are with middle grades teachers as learners, the same careful color-anchoring approach can be profoundly effective with middle and high school learners. The math activities included here were part of a U.S. Department of Education-funded research and professional development project (Hauk & Carroll, 2014). After some background and framing for the use of color in the context of patterns and modeling with mathematics, the bulk of what follows is the description of a modeling activity from the professional development course. The activity was part of the first day and was revisited and referenced throughout the year-long experience. We close with a few thoughts on implementing similar activities in professional development and classroom contexts.

Color in Representing Mathematics

When learners relate parts of a mathematical equation to a graph or word problem, or interpret a diagram, they must direct their attention to each individual source, encode separate pieces of information, and then manage the stored information to make meaningful connections. Paying attention to these multiple sources of information can be challenging. Research indicates that the careful use of color-coding and color-matched text and diagrams improves learning (Sweller, 1994). Furthermore, looking at how others incorporate color into their models provides a window into others' thinking (Friel & Markworth, 2009; Smith, Hillen, & Catania, 2007; Toney, Slaten, Peters, & Hauk, 2013).

How learners make (and make sense of) mathematics through the use of color takes two forms. In investigating students' routes from informal mathematical activity to formal mathematical reasoning, Zandieh and Rasmussen (2010) specify a difference between models *of* mathematical activity and models *for* mathematical reasoning. As we illustrate below, incorporating the use of color into classroom or professional development activities includes attention to color in both the representational role as a model *of* a category or quantity and in the strategic role as a model *for* relationships among ideas and actions.

Illustrating the Ideas: Seats at the Table

The *Seats at the Table* professional development activity is based on a classroom lesson originally adapted from the Mathematics Assessment Resource Service [MARS] (1998; see `http://map.mathshell.org/`) for the mathematics support curriculum *Aim for Algebra* (Gale, 2011)

The purpose of the activity as part of the professional development is as a touchstone example for considering the difference between models *of* and models *for*. That is, we use it to distinguish between *multiple representations* and *multiple strategies*. The structure of the professional development activity provides two opportunities to think about the task. First, participants engage in the activity as learners, considering the mathematical content of the task, noting many possible representations for the situation, and narrating and color-coding their thinking. Along the way, they illustrate for each other a variety of possible solution strategies. Later, participants view the task as teachers, considering pedagogical issues and instructional implications. The task, as presented to the participants-as-learners, shows visual and verbal representations for a patterning scenario (see Figure 1). The scenario involves people who are sitting around a line of contiguous trapezoidal tables. Participants were asked to use these representations, and any others they created, to answer the questions:

(1) How many people can sit at 20 tables? 50 tables?
(2) Find a rule to determine the number of seats at any size table.
(3) How do you know your rule works?

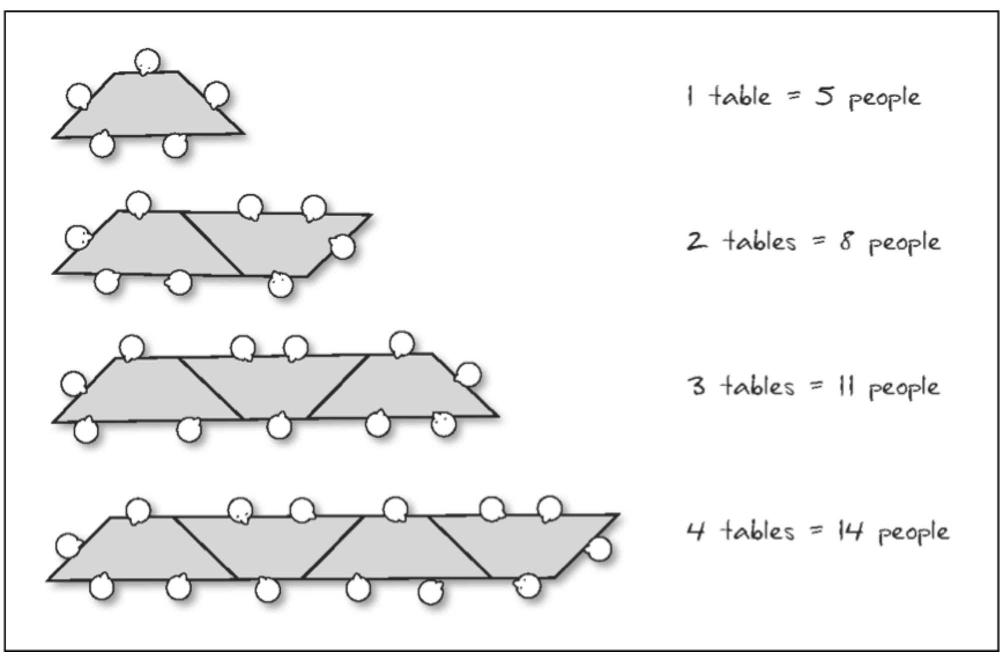
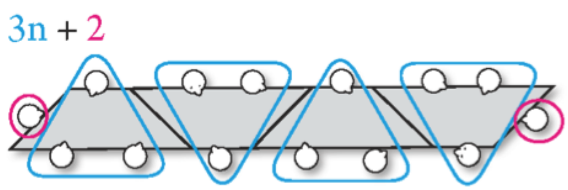


Figure 1. Figural representation in the *Seats at the Table activity*

Participants worked individually, and then in small groups. The facilitator monitored the rules that participants developed, then brought everyone together as a whole group to investigate the different rules and how each rule connected to possible visual and tabular representations.

During their work in small groups, participants rarely used color systematically to support their creation of a rule. After 10 minutes of group work, the facilitator introduced the purposeful use of color to connect representations. In particular, she connected the expression $3n + 2$ and the visual of four tables (see Figure 2). As shown, each group of three at the tables is circled in the same color (blue) and the "3n" is written in blue because $3n$ represents 3 people at each of the n tables. The "2" in the expression represents the 2 people at either end of the row of tables, so those two people in the figure are circled with the same color as is used for the "2" in the expression (here, red). The facilitator also generated lists of values using the same colors (shown to the right in Figure 2) to demonstrate the *development* of the expression $3n + 2$ and its connections to the color-coded sketch of people seated at tables. After this demonstration and a short discussion, participants returned to their small groups to apply the idea of systematic use of color to connect their own rule(s) to the visual for four tables.



# of tables	# of seats	How I saw it
1	5	$3(1) + 2$
2	8	$3(2) + 2$
3	11	$3(3) + 2$
4	14	$3(4) + 2$
...	...	...
n	$3n + 2$	$3n + 2$

FIGURE 2. Using color to show the connection between $3n + 2$ and a figural representation.

Figure 3 (next page) includes four examples of how participants used color and an associated symbolic expression. Each uses color-coded chunks of information that allow comparison across models. Learners could compare the ways of thinking about the problem (multiple strategies) and get insight into a variety of mathematically correct models of the pattern (multiple representations).

Each un-simplified expression, or rule, was a window into a person's thinking. The participant who wrote the rule $4 + 3(n - 2) + 4$ explained the scenario as a number of tables of 3 people in the middle, with a table of 4 people at each end. The participant who wrote $5n - 2(n - 1)$ said that each individual table could seat 5 people, but also noted that seats are lost when the line of tables is contiguous. While both are equivalent to the simplified rule $3n + 2$, they express the constant and variable parts in the linear pattern in distinct ways because the sense-making strategies of the two thinkers were distinct.

Mathematical Ideas and Color Assignment

A process of trial and error, including much discussion among project staff and participants in the professional development, honed our recommendations for the use of color. Ultimately, we determined that assigning a color to each term in the expression (and to where that term is "seen" in the visual) led to productive discussions. Participants' sharing and collective sense-making about the justifications for color associations also side-stepped a possibly counter-productive routinizing of the use of color. We wanted to avoid over-simplifications like "always use blue for the term with n in it" to allow for representations like the third and fourth symbolic expressions in Figure 3 (next page). In each case, the variable n appears in two of the terms.

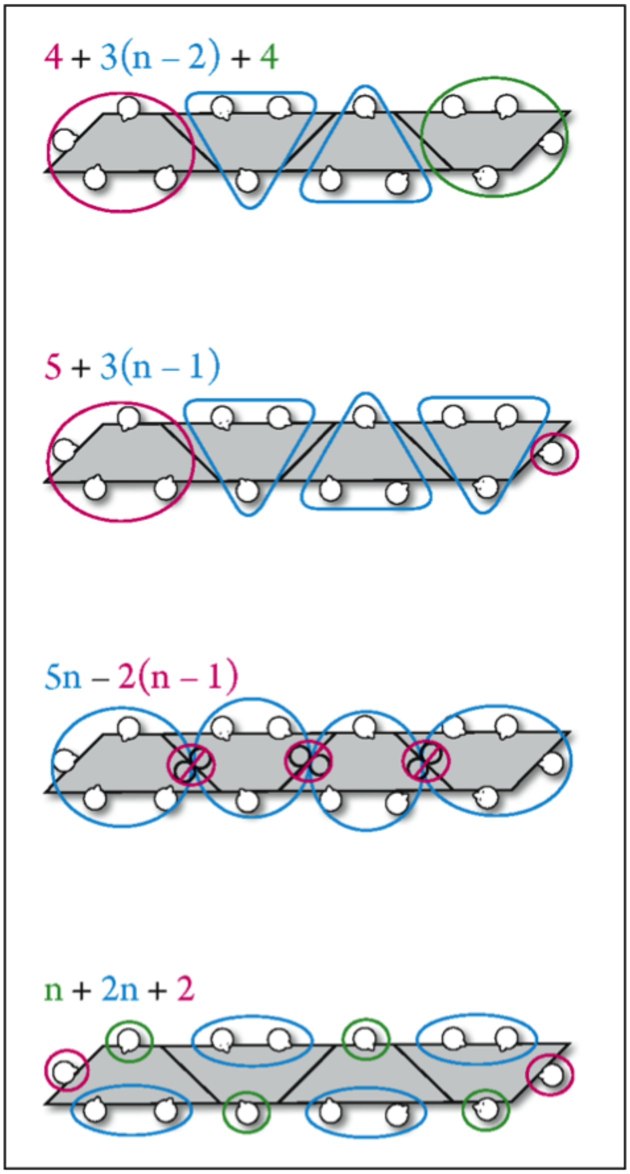


FIGURE 3. Examples of using color to connect symbolic expressions with figural representation for *Seats at the Table*.

Deciding what constituted a "term" in an expression required comparisons across representations and explanations entailed validation of what the group was going to call a "term" in the symbolic representation of the pattern. That is, reflection on the process of coloring their strategies and discussion among participants helped them decide on the details of a socio-mathematical norm for "term." In particular, it led to nuanced consideration of how the model of a particular characteristic of the figure played a role in describing the model for a strategy. For example, the expression $5n - 2(n - 1)$ was a representation that was mathematically equivalent to $n + 2n + 2$, but the color coding and discussion highlighted that it might not be strategically equivalent.

Later, after several tasks that distinguished between the representation(s) of a pattern and the strategies for getting there, participants reconsidered the activity through an instructional lens. They noticed that the elegance of putting a representation in "simplified form" might well obscure valuable information about student thinking processes. This led to conversations about the value of both exploring with representation and practicing with standard forms.

Conclusion

A rich task like *Seats at the Table* allows learners to engage in mathematical practices while also addressing content standards. In the example presented here, we highlighted the practice of modeling with mathematics and focused on the content in the CCSSM Expressions and Equations content domain. Figure 4 summarizes the prerequisite content clusters, target standards, and horizon content clusters (i.e., content that might build on what is addressed in the activity).

Prerequisite standard clusters	
4-OA.C	Generate and analyze patterns.
5-OA.A	Write and interpret numerical expressions.
5-OA.B	Analyze patterns and relationships.
Target standards/clusters	
7-EE.A.2	Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.
7-EE.B	Solve real-life and mathematical problems using numerical and algebraic expressions and equations.
Horizon standard clusters	
A-SSE.A	Interpret the structure of expressions.
A-SSE.B	Write expressions in equivalent forms to solve problems.
A-CED.A	Create equations that describe numbers or relationships.

Figure 4. CCSSM Content Standards Alignment for the *Seats at the Table* activity.

When engaged in mathematical modeling, a learner may use a variety of representations: rule, expression, visual, table, and so on. Noticing the similarities and differences in un-simplified models of a pattern (multiple representations provided by multiple thinkers) offers insight into the variety in valid strategies for modeling a scenario. The use of color can highlight connections among elements in the various representations and allow comparison across different strategies. Taking time to examine the different ways of modeling a problem supports teachers and students in building a repertoire of ways to view patterns.

The participants who were part of the professional development agreed that using color to model patterns and algebraic expressions had been very powerful for themselves and for their students. At the end of the year, one participant explained it this way:

> *The use of color with patterns and algebraic equations: that tiny little thing just really transformed my algebra unit this year, and I feel like they understand all those concepts so much better. So I think for them and for me, that's a huge takeaway.*

Though we focused here on the context of professional development, the activity can be scaffolded in the classroom for use with students. Key in the scaffolding are (1) asking learners to make sense of the situation first as individuals and in small groups *before* (2) offering a demonstration of a particular systematic use of color, then (3) follow with time for learners to re-cast their initial efforts using the newly introduced color-coding and (4) whole group time for people to describe how their particular strategy is rendered through color and (5) reflections by learners on things to remember about the process and connections to "simplified" expressions. While other ways for incorporating color into representations could be appropriate, what is important is that the use be consistent across associated representations.

Further evidence about the potential effectiveness of modeling in color as part of professional development and instruction is emerging in early results from a related study of middle school student learning outcomes. These indicate that students in the classes of the project's teacher participants had greater gains in understanding of figural representations, tabular representations, and translation across representations in pattern problems than comparison students in the classes of non-participating teachers.

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References

Common Core Standards Writing Team. (2015, May). *Progressions for the common core state standards in mathematics* (Tech. Rep.). Tucson, AZ: Institute for Mathematics and Education, University of Arizona. Retrieved from `http://ime.math.arizona.edu/progressions`

Common core state standards for mathematics. (2010). Washington, DC: Council of Chief State School Officers [CCSSO].

Friel, S. N., & Markworth, K. A. (2009). A framework for analyzing geometric pattern tasks. *Mathematics Teaching in the Middle School*, *15*(1), 24–33.

Gale, M. (2011). *Aim for algebra: Patterns.* Mount Kisco, NY: It's About Time.

Hauk, S., & Carroll, C. (2014). *Making Middle School Mathematics Accessible to All Students.* U.S. Department of Education, R305A100454.

Schifter, D., Bastable, V., & Russell, S. J. (2009). *Patterns, functions and change: Facilitator guide.* Upper Saddle River, NJ: Dale Seymour Publications.

Seago, N., Mumme, J., & Branca, N. (2004). *Learning and teaching linear functions.* Portsmouth, NH: Heinemann.

Smith, A. F., Hillen, M. S., & Catania, C. L. (2007). Using pattern tasks to develop mathematical understanding and set classroom norms. *Mathematics Teaching in the Middle School*, *13*(1), 38-44.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, *4*, 295-312.

Toney, A. F., Slaten, K. M., Peters, E. F., & Hauk, S. (2013). Color work to enhance proof-writing in geometry. *Journal of the California Mathematics Project*, *6*, 9-20.

Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *Journal of Mathematical Behavior*, *29*, 57–75.

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