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An Amazing, Space Filling, Non-regular Tetrahedron: As Simple as $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$

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Abstract. *Preface for the Teacher:* Exploring the shapes in this paper can increase student understanding of basic concepts of space geometry and allow teachers and students to consider ideas such as these:

- How can cubes be examined and used to create other examples of polyhedra that fill space?
- How can relationships from plane geometry such as the Pythagorean theorem, be used to find lengths and shapes to create nets to build space-filling polyhedra?
- How can models of these space-filling shapes be constructed and used to build intriguing, hinged 3-dimensional puzzles?

1. Introduction

In the plane, three regular polygons, the square, the equilateral triangle and the regular hexagon, can be arranged to tile the plane without gaps between the shapes. In space, as you know from building with blocks, the cube can fill or tile space in the same way. Filling or tessellating is distinct from packing space. Space packing refers to an arrangement of objects in three-space. The objects touch in some specified way, but voids are allowed to exist. In a space filling or tessellation, there are no voids. In *On the Heavens*, written in 350 B.C., Aristotle proclaimed that the regular tetrahedron fills space with no gaps. He was mistaken.

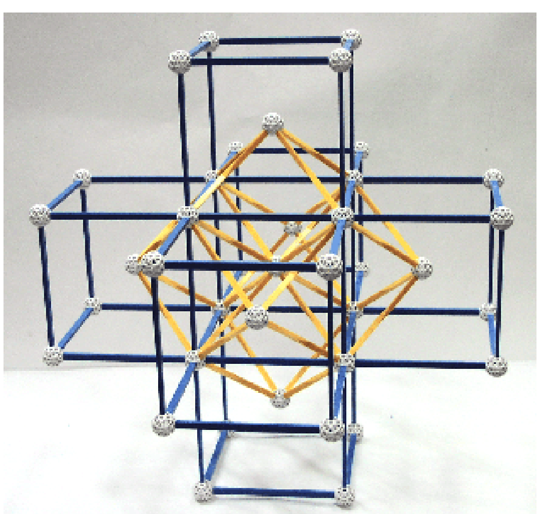

Figure 1

However, a particular tetrahedron, with dimensions derived from the cube, does fill space (Frost & Koch, 2005). This tetrahedron with its closely related polyhedra — the cube and the rhombic dodecahedron — and their space filling properties will be the focus of this paper.

Figure 1 illustrates what will follow. It shows a rhombic dodecahedron sitting inside a framework of cubes. This shape derives its name from its twelve (dodeca) faces, all of which are congruent rhombi. This paper will explore the connections between a special non-regular tetrahedron, the cube, and the rhombic dodecahedron, and will show how to use their common dimensions to create a net or pattern to build three intriguing puzzles.

2. Filling the Plane Using a Checkerboard

Before moving into three-space, let's look at the idea of filling the plane in two dimensions. The square is one of the polygons that clearly will fill or tessellate the plane.

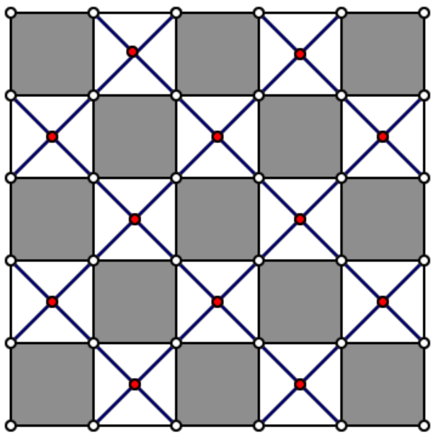

Figure 2

Imagine that you are looking at an infinite checkerboard of alternating black and white squares. Each white square has a our neighboring black squares. For any point in a white square, you can ask which of the black squares is closest. Then collect together all the white points that are nearest to each of these black squares.

This is accomplished by dissecting each of the white squares along its two diagonals, forming four congruent triangles, each one sharing an edge with a black square (Figure 2). The points on the diagonals are an equal distance from two or more black squares.

Now imagine "re-assigning" points into new shapes by attaching each white point to its nearest black square. Each triangle is then added onto the black square with which it shares an edge. This divides the plane up in a different way so that it is covered by pieces that look like a black square with four white right triangles attached (Figure 3).

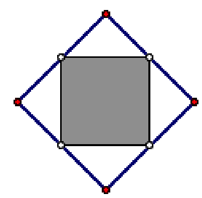

Figure 3

By examining this figure, you can see that the new shape is also a square, which is twice the area of the original square, and rotated 45 degrees. The plane is covered by these new larger squares.

3. Filling Space Using a 3D "Checkerboard"

Now extend this exercise into three-space. The cube is one of the polyhedra that will fill or tessellate space. In the three-dimensional case, the "checkerboard" is made of alternating black and white cubes. As you did before, you can re-assign the points in each white cube so that each point is attached to the nearest adjacent black cube. This is accomplished by dissecting the white cube into six congruent square-based pyramids formed by the space diagonals of the cube. One of the pyramids is shown in Figure 4. Each pyramid is then added onto the black cube with which it shares a face. The resulting shape is made of the black cube and six square pyramids.

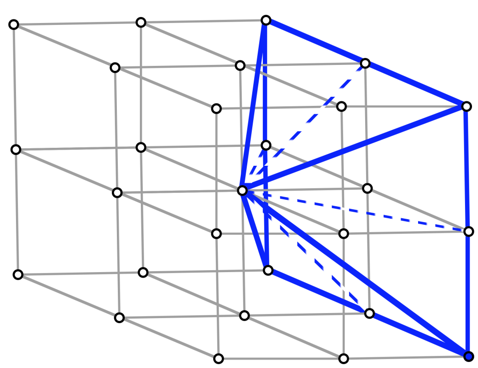

Figure 4

By counting, this new shape would seem to have four triangular faces for each of the six pyramids, for a total of 24 triangles. But as in the two-dimensional case, pairs of triangles from neighboring pyramids fit together to form a planar quadrilateral face—a rhombus but not a square. So, the polyhedron has twelve rhombi as faces. This is a **rhombic dodecahedron**.

Since the black and white cubes filled space, this new arrangement of rhombic dodecahedra also fills or tessellates space, with no gaps between. Each rhombic dodecahedron has the volume of a black cube and a white cube, or twice the volume of one of the original cubes.

You can see the reason the triangular faces of the pyramids fit together to form a planar figure by adding the dihedral angles between the planes of the polyhedra used to construct the rhombic dodecahedron. Since the planes of the triangular pyramid faces bisect the 90-degree angles between the faces of the cube, the triangular faces of a pyramid meet their base at a 45-degree angle.

Visualizing a pyramid added to the side of a cube and an additional pyramid added to the top of a cube, the triangular faces of the pyramids that meet at the edge of the cube are the faces you want to demonstrate are co-planar. Adding two angles of 45 degrees with the right angle between the faces of the cube gives $45\,\text{deg} + 90\,\text{deg} + 45\,\text{deg} = 180\,\text{deg}$, so that the angle between a pair of adjacent triangular faces is 180 degrees. This demonstrates that the triangles lie in the same plane.

4. Triangles with Sides $\sqrt{1}, \sqrt{2}, \sqrt{3}$ are Basic Building Blocks

When children play with blocks, constructing towers and walls with them, they exploit the space filling, or tessellating, properties of the cube. Nature provides marvelous examples of this shape found in crystals of iron pyrite, fluoride, halite (or salt), galena, and the like. Because the cube is so familiar, it is a perfect place to start when investigating space filling with students.

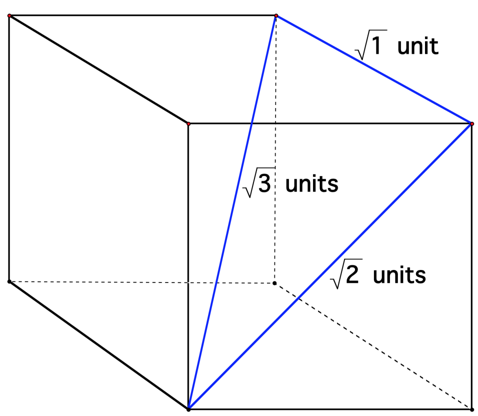

Figure 5

While the cube is very familiar to students, they seldom realize that square roots are necessary to investigate the relationships among the edges and the diagonals of the cube. Even those students who are proficient at using the Pythagorean theorem to calculate the missing edge length of a right triangle, may not have encountered its application in three space. With some gentle guidance, most students will be able to visualize the triangle in space whose edges are the edge of the cube, the face diagonal (the diagonal on one of the square faces) and the space diagonal (the diagonal connecting two opposite vertices, and passing through the interior of the cube) in Figure 5.

If the side of the cube is s, the face diagonal can be calculated by applying the Pythagorean theorem to a right triangle that is one-half of a square face to get the length $s\sqrt{2}$. Next, this result can be used to calculate the space diagonal. Challenge your students to generate a direct means for calculating the space diagonal without the intermediate step. Using this three dimensional version of the Pythagorean theorem, they can show that the space diagonal for any rectangular prism is equal to $\sqrt{(a^2 + b^2 + c^2)}$ where a, b and c are the dimensions of the sides of the prism. When applied to a cube, the space diagonal is equal to $s\sqrt{3}$, where s is the edge length of the cube.

The lengths of the edges of the triangle, if the side of the square is s (which can also be written as $s\sqrt{1}$), are $s\sqrt{1}$, $s\sqrt{2}$, and $s\sqrt{3}$. This "$\sqrt{1}, \sqrt{2}, \sqrt{3}$ triangle" is a basic shape that appears in the cube, the rhombic dodecahedron, the first stellation of the rhombic dodecahedron, and the tetrahedron being examined in this paper.

This triangle also can be seen as half of a rectangle. Start with the rectangular cross-section of the cube cut by a plane through opposite edges AB and $A'B'$ of the cube (Figure 6a). This rectangle (Figure 6b) is a rectangle whose diagonals are two of the space diagonals AA' and BB' of the cube. The rectangle has sides s and, $s\sqrt{2}$; so the diagonal AA' has length $= s\sqrt{3}$.

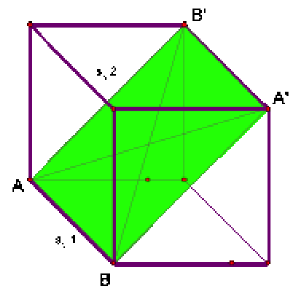

Figure 6a

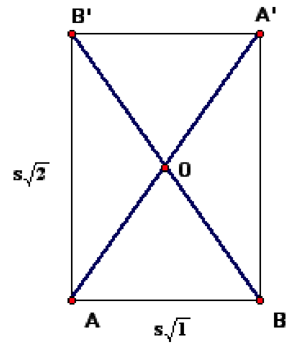

Figure 6b

5. Puzzling Isosceles Triangles

In the rectangle shown in Figure 6b, the isosceles triangle ABO is one of the triangular faces of the six square pyramids formed from the cube. Two of these isosceles triangles fit together to form a rhombus that is a face of a rhombic dodecahedron.

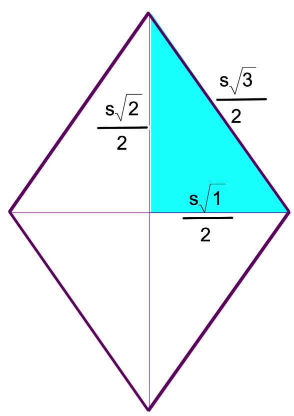

Figure 7

This isosceles triangle will be fundamental to the construction of three-dimensional puzzles for the classroom activity, so the working name for the purposes of this paper is **"puzzling" isosceles triangle**.

If the triangle ABO is cut into two right triangles by an altitude through O, the sides of the right triangle are $\frac{1}{2}s\sqrt{1}$, $\frac{1}{2}s\sqrt{2}$, $\frac{1}{2}s\sqrt{3}$, so this triangle is similar to the triangle with sides $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$.

Thus, the puzzling isosceles triangle is made from two triangles similar to the $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$ triangle, and the rhombic face of the rhombic dodecahedron is made from four such triangles (Figure 7).

5.1. Constructing the puzzling isosceles triangle. The puzzling isosceles triangle can be constructed using a compass and straight edge, geometry software such as *The Geometer's Sketchpad*, or using simple paper folding techniques.

Paper folding to construct the basic triangle followed by reflection to construct the rhombus is an option accessible to even young students. Students in an algebra or geometry course can use the Pythagorean theorem and reflection to generate the rhombus based on the cube's edge length. Students in a trigonometry or calculus course can use the inverse trigonometric functions to build the face based on the edge lengths and angles. When approached trigonometrically, the angles can be calculated with the inverse functions. The larger angle in the rhombus is double the larger acute angle in the $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$ right triangle. Using right triangle trigonometry, this rhombus angle equals $2\tan^{-1}(\sqrt{2})$, which is approximately 109.47 degrees. The smaller vertex angle of the rhombus is supplementary to this angle or 70.53 degrees.

5.2. Directions to paper-fold the puzzling isosceles triangle.

The triangle can easily be created by folding an 8.5 x 11-inch piece of paper. Have students turn the paper so the longer side is horizontal. This side will be the base of the triangle. Bisect the base by folding the paper in half. This creates a vertical center fold and two 90-degree angles. Next bisect either right angle to produce a 45-degree angle by folding the bottom vertex onto the altitude, see Figure 9a.

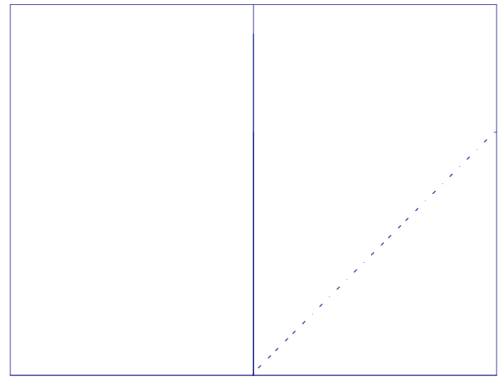

Figure 9a

Imagine the square for which this new fold is the diagonal. This length will be the height of the triangle. Bisect the 45-degree angle again, to transfer the length of the folded edge to the center fold line (Figures 9b and 9c). Place a dot at the point where the diagonal of the square lies along the center fold. This point is the third vertex point of the triangle.

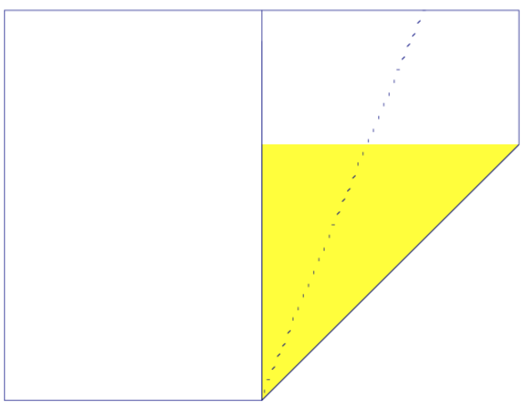

Figure 9b

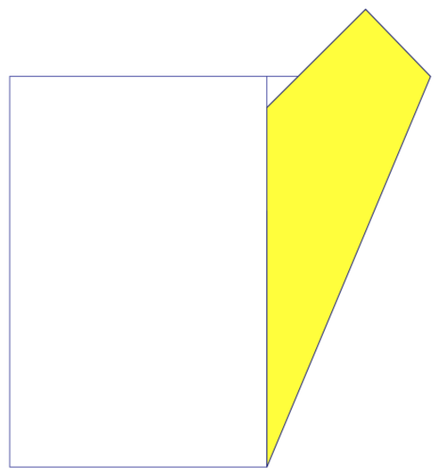

Figure 9c

Unfold the paper and connect this top vertex point with each of the two bottom corners of the paper. This triangle has base length of 2 units and a height of $\sqrt{2}$ units, Figure 9d. Check that this is, in fact, the puzzling isosceles triangle. Verify the lengths of the outside edges and the measures of the three angles.

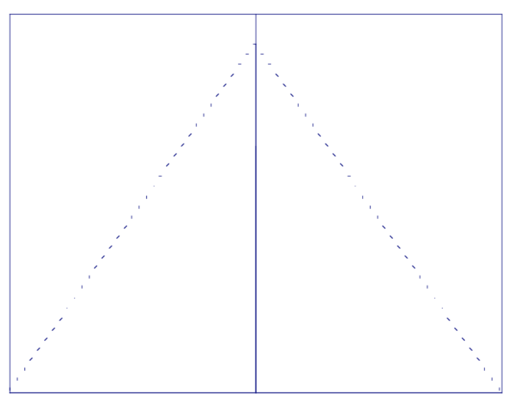

Figure 9d

6. Building a Tetrahedral 3D Puzzle Piece

Cutting the rhombic dodecahedron into special irregular tetrahedral can make fascinating three-dimensional puzzles. In fact, the net for the tetrahedron puzzle piece can be made from the puzzling isosceles triangle shown above. The net for the tetrahedron is made up of the four midpoint triangles of the puzzling isosceles triangle.

6.1. Directions to paper fold the net for the tetrahedron. The net for a tetrahedron is made of four triangles. This net is made from the puzzling triangle shown in Figure 9d by dividing the triangle into four congruent triangles, each one similar to the original triangle. Continuing with the triangle in Figure 9d, bisect the two remaining side lengths by placing the right vertex directly on the top vertex and making a crease for the midpoint. Repeat this step with the left vertex positioned on top of the top vertex (Figure 10a). These midpoints can also be found by bringing the top vertex of the large triangle to the bottom edge, being sure the original vertical fold lies along itself.

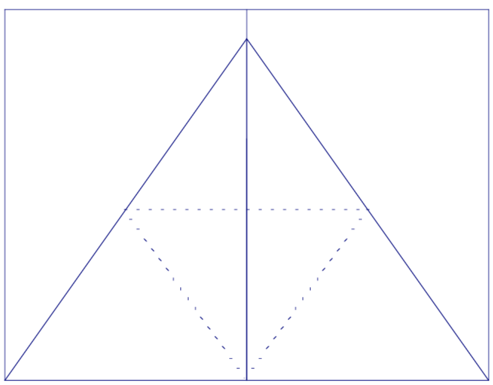

Figure 10a

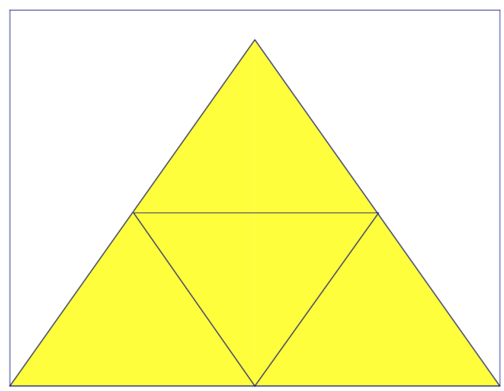

Figure 10b

Using a pen and straight edge, connect each of the midpoints of the three sides (Figure 10b). The original lengths are all cut in half, so each of the four triangles is similar to the puzzling isosceles triangle. How can you prove this?

You now have a net for a special tetrahedron that will be used to create three related puzzles.

6.2. Space-filling properties of the special "puzzle" tetrahedron. Now that you have created the net for this tetrahedron, examine both its special space filling properties and its relationship to the rhombic dodecahedron. The following are suggested areas of exploration and discussion to consider pursuing with your students:

- At two of the edges, the faces of this tetrahedron meet at 90-degree angles. Ask students to explain this by mentally cutting the tetrahedron in half and using the Pythagorean theorem. (Hint: The altitudes of the two faces forming the angle are both $\frac{1}{2}\sqrt{2}$. These two segments are two sides of a triangle in space with the third side being an edge of the tetrahedron. It is possible to show that this triangle is a right angle and that the angle of the triangle is the (dihedral) angle between the faces.)

- Ask your students to show how to use this right angle between planes to fit two tetrahedra together so that two faces join to form one of the rhombic faces of the rhombic dodecahedron.
- Ask your students to show how to use this right angle between planes to arrange four of these tetrahedra to form a double pyramid made of two of the pyramids studied above.
- Ask your students to explain how to fit together a number of these double pyramids to form a rhombic dodecahedron.

6.3. Seeing the special tetrahedron in the photo of a rhombic dodecahedron. As you calculated above, twenty-four of these units can pack around a central point to create a rhombic dodecahedron. The picture in Figure 11 is the photo of a puzzle made of black and white "puzzle" tetrahedra whose net was constructed above. It is challenging to visualize the shape from the photo.

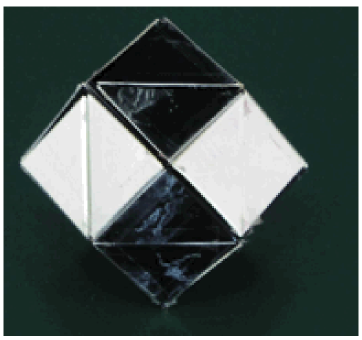

Figure 11

First, notice that four rhombic faces are visible. Each rhombus is formed from two triangles. As you saw above, each triangle is one of the puzzling isosceles triangles that are faces of the puzzle tetrahedron. This fits with the observation in the previous section that two tetrahedra fit together to form a rhombic face.

Second, notice the square pyramid in the center with four triangular faces. This is the visible portion of the square double pyramid made of four puzzle tetrahedra. As you examine the picture, visualize the cube hidden inside, surrounded on each face by a square pyramid.

Also, notice how much the photo resembles the plane checkerboard figure at the beginning of this paper. This helps to illustrate how an infinite number of these rhombic dodecahedra would fill or tessellate space.

6.4. Seeing the special tetrahedron in the photo of a stellated rhombic dodecahedron. The photo in Figure 12 shows a solid derived from the rhombic dodecahedron by positioning two special puzzle tetrahedra on each of the twelve faces of the rhombic dodecahedron. Another way to create this shape is by extending each of the faces of the rhombic dodecahedron outward until they intersect, forming what is called the first *stellation* (stellation meaning star) of the rhombic dodecahedron. What a beautiful shape!

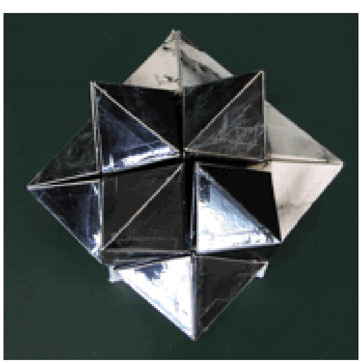

Figure 12

If you are familiar with M. C. Escher's print "Waterfall," this shape is atop the tower in the upper right hand corner.

This shape also fills (tessellates) space. This is not easy to see, but here are some hints. Imagine slicing the shape into eight congruent pieces by cutting the shape along three perpendicular planes (forming eight octants in space). One plane should be parallel to the picture plane (slicing through the square you see looking at it from above) and the others should be perpendicular to the picture plane, to each other, and to the edges of the square that is the projection on the picture plane (as in Figure 13).

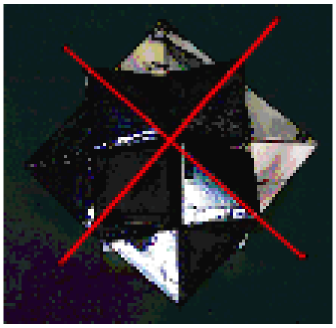

Figure 13

Each of the eight pieces is half of a cube with squares on three faces. By surrounding the figure above with a jacket of eight more of these half cubes, the new polyhedra will be a larger cube. These half cubes are made up of three square pyramids.

This is difficult to visualize without physical models, but it does explain how to fit stellated rhombic dodecahedra together to tessellate space. Try to convince yourself that an infinite number of these really would fill space in the same way that a cube fills space. You may need to make up four to six (or more) of these and try stacking them together.

7. Building Puzzles Using the Tetrahedra

As a class activity, have your students create the puzzles illustrated in the pictures included in this paper. Working with these puzzles gives students a context to use and apply important geometric vocabulary and concepts they have learned. These puzzles also encourage perseverance and quality work from students as they discover that care and exactness in construction allows the puzzles to fit together tightly and move smoothly.

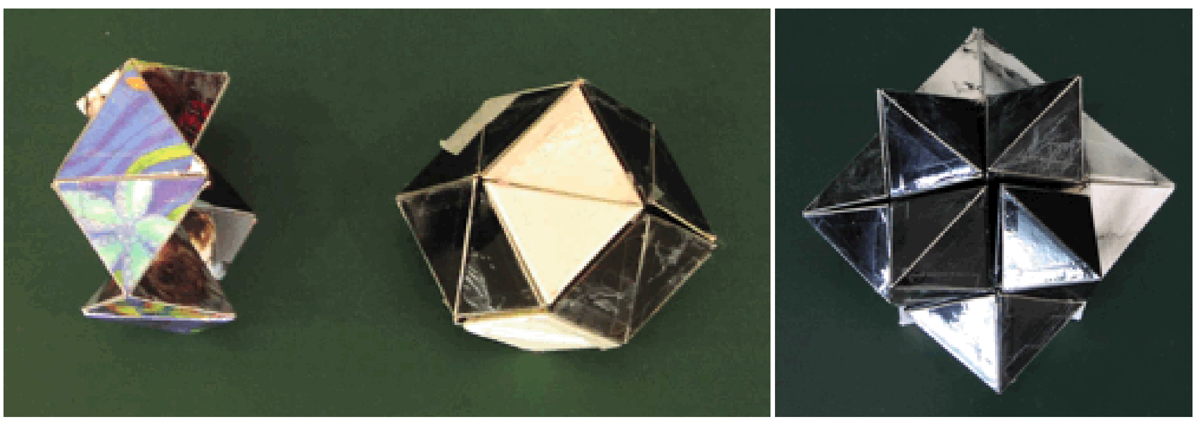

Figure 14. Rotating Ring (left) Rhombic Dodecahedron (center) and Stellated Rhombic Dodecahedron (right).

7.1. Models from manila file folders. After each student has carefully folded the tetrahedral unit (see Figures 9–10) using an 8.5 by 11-inch piece of paper, have them glue-stick their paper to half of a manila file folder and cut out the large triangle. Using a straight edge and a sharp pen (e.g., a roller ball pen), they should line up the straight edge next to each of the three mid-segments and run the pen along the line. This will produce a sharp crease along the line. To make this process (called scoring) even more effective, they should place about five sheets of paper under the file folder to act as padding. Identify the long edges of the triangle for later use.

The triangle is then folded into a tetrahedron and taped along the three remaining edges. Care should be taken to be as accurate and careful as possible with these units since the puzzle should pack together fairly tightly. Packing tape works best for taping the tetrahedra because it will not split and is much less expensive than Scotch or Magic tape.

7.2. Make a basic pair of tetrahedra. As you look at each tetrahedral unit, pick out which edge length has a length of two units and which edge length is $\sqrt{3}$ units (about 1.73 units). As you place two units facing each other (mirror images), the longer edge (represented by the dashed line in Figure 15) will create the connection making a flat rhombus sitting on the table as shown in Figure 15. The bold edges rising above the table will be the shorter lengths ($\sqrt{3}$ units). The outside edges (touching the table) will also be $\sqrt{3}$ units. The two remaining edges (rising above the table) will be the longer edges like the common edge on the table (dashed line).

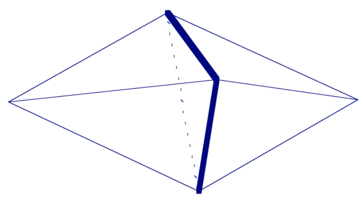

Figure 15

Hinge this basic pair with tape along the common edge on the table. Then bend the pair at this hinge so that you can tape the hinge on the other side with a second piece of tape. Double taping the hinges in this way is important to keep them from falling apart.

Model 1: Rotating ring from 8 units. To create a rotating ring of tetrahedra, eight of these tetrahedral units need to be connected along their long edges using strips of packing tape. Make up four sets of basic pairs of tetrahedral units (as above). Hinge these sets together along their long edges (tape the hinges again on both sides). This creates a hinge between each basic pair of rhombus-faced pieces. When all the pieces are connected together in a tight ring with the shapes meeting together in the middle, they pass each other as they rotate around. The rotating ring is on the left in the picture below. This is one of the kaleidocycles in Doris Schattschneider and Wallace Walker's (1987) book, *M. C. Escher Kaleidocycles.*

Model 2: Rhombic dodecahedron from 24 units. By connecting twenty-four of the tetrahedral units along their shorter lengths (thicker lines in the sketch) you can create a rhombic dodecahedron. You can also form this shape by taping along their longer lengths. In either case, this creates a long ring that looks much like a necklace that can be pulled together into the shape of a rhombic dodecahedron.

Model 3: Stellated rhombic dodecahedron from 48 units. George Escher (personal letter, September 6, 2001) stated that forty-eight of these tetrahedral units can be arranged to create the stellated rhombic dodecahedron. However, the units need to be connected in a ring and hinged along their shorter lengths to make this work. Have each student create a tetrahedral unit cut from 8.5 by 11-inch paper and glued to a file folder. The resulting chain will be about 15 feet long and makes quite a challenging puzzle to put together. If you plan to keep it together as a stellated rhombic dodecahedron, you will need rubber bands to secure the puzzle.

Acknowledgments

The work reported was shared through the Park City Math Institute.

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About the Authors

Joyce Frost has had a long career as a junior high school and high school mathematics teacher, teaching all levels of mathematics through AP Calculus. This is particularly amazing since she first graduated with a Choral and General Music degree with the intention of teaching music. After picking up a 7th grade math class her first year of teaching, she switched from working on a master's degree in Music Education to a Mathematics degree. It was a very good thing because she met her husband, Joe, in Linear Algebra. Both Joyce and Joe earned 2nd degrees in Mathematics and have been collaborating on math ever since. An avid Escher fan, Joyce attended the 1998 MC Escher Congress in Rome, Italy and started a 20-year correspondence with George Escher, MC Escher's eldest son. Through the years, they shared the creation and invention of 3-D puzzles with each other. This work inspired the puzzles that are the basis for two of her three articles published through the Park City Math Institute (PCMI) Teacher Program. Joyce has been a PCMI participant and staff member. She continues to put together programs for the Puget Sound Council of Teachers of Mathematics, the Northwest Mathematics Interaction, and Northwest Mathematics Conferences including the 2007, 2013, and 2019 conferences. In her spare time, she travels with the Seattle Tacoma Friendship Force Club, plays hand bells and sings in her church choir, tutors math students, and spends time with her grandchildren, two of whom live in Basel, Switzerland.

Margaret (Peg) Cagle is a Presidential Awardee for Excellence in Mathematics Teaching, Albert Einstein Distinguished Educator Fellow, and former board member of the National Council of Teachers of Mathematics. Peg also was a practice faculty member at Vanderbilt University. She currently teaches high school in Los Angeles, and serves as a Director of the Park City Math Institute/Teacher Leadership Program, and is an NSTA/NCTM STEM Teacher Ambassador. Peg is a registered architect, enjoys exploring the intersection of mathematics and traditional fiber arts, and has the largest known privately held collection of Zometools in the universe.