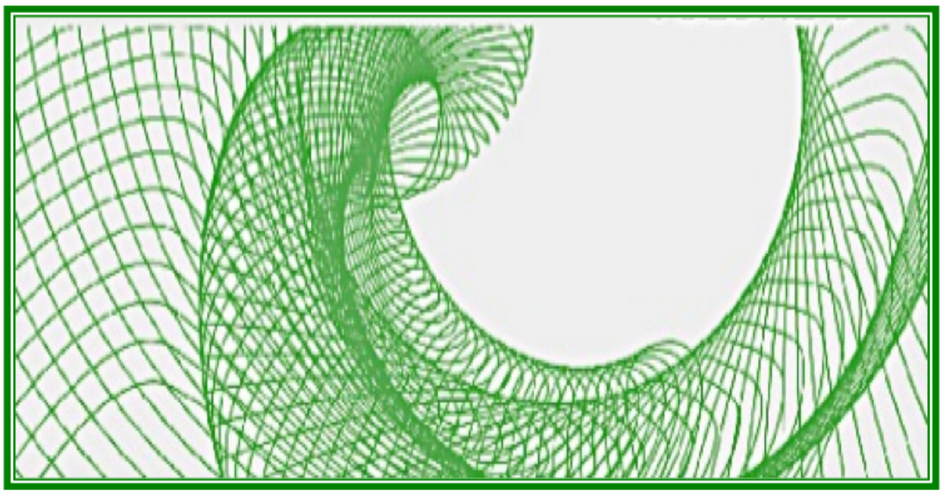
JOURNAL OF THE CALIFORNIA MATHEMATICS PROJECT



Volume 11 Fall 2022

Introduction

The *Journal of the California Mathematics Project* (*JCMP*) is a publication of the California Mathematics Project (CMP) and is sponsored by San Francisco State University and California State University Stanislaus. The journal supplements the official news publication of the CMP, *California Online Mathematics Education Times* (COMET), ⟨https://cmpso.org/comet/⟩.

The journal's mission is communication about mathematics education among those engaged in it – through the CMP or similar initiatives anywhere. Authors include K-12 teachers, higher education faculty, and others involved with mathematics education, such as graduate students and school leaders. The call for submissions is on-going. We do accept simultaneous submissions (copyright is retained by the author). If *JCMP* is the first to accept an article for publication, the *JCMP* publication must be cited in all other publications of that article, even in revised form.

As can be seen in this issue, the journal publishes a wide variety of submissions, including brief research and research-to-practice articles, reports of lessons and classroom practice, and book reviews. We also welcome reviews of state adopted materials and programs from authors who have experience using them.

Submission and Review of Material for Publication

Manuscripts are accepted in .rtf, .doc/.docx, and .tex formats, using 12 point Times New Roman or a font with similar size and spacing and with 1.25 inch margins on all sides. Articles are published with LaTeX(a production editor works with authors on formatting). List references at the end of the article in alphabetical order by first author last name with appropriate corresponding citations in the text in American Psychological Association (APA) 7 style. See the articles in this volume for examples of appropriate style and length. For more information and the electronic submission website, please see: https://jcmp.calstate.edu

Submissions will be refereed and may be accepted or returned for revision. Initially, we use a double blind consideration process. To help prepare an article for publication, a referee may reach out to the author to suggest improvement.

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Table of Contents

Journal of the California Mathematics Project
Volume **11**, 2022

By Inches and Yards
In Honor of Viji Sundar

Bob Stein, Susie W. Håkansson, Carol Fry Bohlin, Veronica Chaidez, Angelo Segalla, Shandy Hauk, and Mark K. Davis

Abstract. The mathematics education community was saddened by the passing of its eminent member, Viji Sundar (18 March 1943 – 17 November 2021). The Editors invited some of the many who worked with Viji over the years to share stories. In celebration of a life well-lived and a legacy with staying power the Dr. Viji Sundar Memorial Fund <https://stanforacause.csustan.edu/project/29635> was established to support the ongoing influence of her passion for learning.

Born in Kerala, India, Viji Sundar completed a bachelor's degree in the department of Mathematics, Physics and Chemistry at the University of Poona, India. She went on to earn both a master of science and doctorate in mathematics at the University of Illinois, Urbana-Champaign.
In 1978, she joined the faculty at California State University Stanislaus (Stan State). In the ensuing 42 years, in addition to becoming a full professor (1988) she served the students and department in many ways (including as department chair, 2005-2008). She also served the region and state as the director of the Central California Mathematics Project (1993-2016). Dr. Sundar's contributions to teaching, learning, and supporting excellence in science, technology, engineering, and mathematics education were acknowledged by many awards. Among these were the Wang Family Excellence Award and three Outstanding Professor awards at Stan State as well as distinguished service awards from outside the university, including recognition by the Stanislaus County Commission as one of 2008's Outstanding Women and the 2015 Distinguished Service Award from the San Joaquin Engineers Council.

I Said Yes

Bob Stein, Editor-in-Chief, Journal of the California Mathematics Project (JCMP)

Viji Sundar was the animating force behind the creation of this journal. Although a tiny woman physically, her passionate devotion to mathematics education and to helping women in the field were colossal. I knew her first from statewide meetings of the California Association of Mathematics Teacher Educators (CAMTE) then from lecturing a couple of times as a guest at her invitation. I was so impressed by her enthusiasm for seizing opportunities that when she invited me to serve as editor of JCMP I quickly agreed.

Viji envisioned this journal as providing opportunities for people working in mathematics education, particularly those early in their careers, to publish their work and, more generally, for sharing ideas and experiences about mathematics education. I am proud to remember Viji and to carry on, devoted to her spirit and ideals.

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Memories of Viji Sundar

Susie W. Håkansson

Dr. Viji Sundar was the Site Director and Faculty Advisor of the Central California Mathematics Project (CCMP) since its inception in 1984. Although I interacted with her for many years as a regional CMP site director, it wasn't until 1999 when I became the CMP Executive Director, that I observed and experienced the thoughtful, kind, and caring human being she was. Since we are both short, we connected immediately. I visited her summer institutes for many years and observed her expertise and compassion as she worked with teachers in the teaching and learning of mathematics. Through her guidance, CCMP provided a variety of professional development programs to increase teachers' effectiveness in the classroom. In many of the CCMP summer programs, teachers from Thailand were invited to participate. This partnership with Thailand broadened the perspectives on teaching and learning of mathematics for all teachers.

In 2009, CCMP established the *Journal of the Central California Mathematics Project* to share with the broader mathematics education community what had worked for them. In my introduction to the first issue of the journal, I wrote the following:

> To prepare for the writing of this journal, CCMP held a summer institute during 2008 for participants interested in writing articles for this journal. Consultants with expertise in mathematics and/or writing supported the participants, reading many drafts and providing technical assistance. I had the privilege of visiting CCMP during this institute and fit right in since I had a writing task to complete that week. It was great to see the participants so engaged in their writing efforts, so much so that they were often oblivious to their surroundings.

Viji provided the opportunity for those who were interested in writing to realize their passion. She also provided the venue to share the best practices that her teachers had found. Subsequent to that initial volume in 2009, in 2012 the publication became the Journal of the California Mathematics Project. Viji's vision of communicating what is effective in teaching mathematics continues to be realized as evidenced by the continuation of the journal.

Viji provided an opportunity for me in 2001. She invited me to accompany her to visit the mathematics educators from Thailand who had attended the CCMP summer institute. I was excited to have the opportunity to interact with the Thai mathematics educators in their environment. However, that was the year of 9/11, so we decided not to go in November. Unfortunately, when Viji rescheduled, I was not able to go. She was so thoughtful and brought back a shoulder padding for a seat belt from Thailand, which I still use to this day.

She also knew how to celebrate and have fun. She invited me to her daughter's wedding that was held in the Los Angeles area. I remember the groom coming to the wedding on a horse, the elaborate ceremony, and the feast afterwards. What I experienced was a meaningful celebration embracing attendees as an extended family and sharing in the joy.

Viji cared about others, She cared about all the teachers she worked with. She cared about her students at CSU Stanislaus. She empowered her teachers and students to communicate their successes so that others may also experience success. It has been my privilege to have known and worked with Viji. May her vision of sharing what works carry on.

Reflecting on the Life of Dr. Viji Sundar

Carol Fry Bohlin

The incomparable Viji Sundar will certainly live on in the memories of those who were fortunate enough to have crossed her dynamic path. California Mathematics Project (CMP) site leaders who learned of her passing during the CMP Directors Meeting in December 2021 were stunned and saddened, but we found comfort among others within that special professional family in which Viji had served as the influential and longtime Director (1993-2016) of the Central California Mathematics Project based at California State University, Stanislaus.

While short in stature, Viji was a giant in her impact. She was passionate about empowering, supporting, mentoring, and celebrating her students and her colleagues. She used her gentle but persuasive powers to motivate individuals to contribute their time and talents to efforts that served students of all ages and the professional community. The *Journal of the California Mathematics Project*, the High School Mathematics Access Program (HiMAP), Preparing Women for Mathematical Modeling and Robotics (PWMM-R), and the Sundar Institute are just a few efforts that benefitted from her leadership.

Viji stepped forward when there was a call for candidates to run for secretary of the California Association of Mathematics Teacher Educators (CAMTE). In addition to serving as secretary, she also served on CAMTE's Advocacy Committee and presented on preparing future middle school mathematics teachers within the CAMTE strand at two California Mathematics Council – North conferences.

Although Viji was always actively involved professionally, she never lost sight of the human side of life and cultivated friendships with care. She often surprised students and friends with thoughtful gifts. After I gave birth to my son, Viji brought two gifts to the next CMP Directors Meeting, a book for Christopher and a bracelet for me. As she held my hands in hers and looked into my eyes, she said with sweet conviction that a new mother should be celebrated as much as her child, yet this is too rarely done.

That was Viji – always teaching, always challenging, always inspiring! May her memory continue to inspire all who knew and loved her!

Standing in Dr. Viji Sundar's Shadow

Veronica Chaidez

I once self-identified as the epitome of an at-risk youth. I unapologetically claimed such title as a means to justify and make sense of my life. I was born into a low-income minority family who did not push an educational agenda. Out of respect, I must note that my parents, through no fault of their own, did not understand the value of an education since elementary was the highest educational level they had achieved. I grew up in a high poverty and high crime neighborhood with little prospects. Recognizing that I had limited options and no hope for a bright future, I became involved with gangs and gang activity before the age of 13.

I navigated through life numb, recklessly, and without purpose. However, my most cherished memory, during this time, was and is when I became a teenage mom, to a healthy baby, at the age of 14. I remember being so scared of giving birth. Not because of the pain of childbirth. Not because I was a child giving birth to another child but because of a more sinister reason. At the time when I became pregnant, I was still affiliated with a gang. Once they discovered I was pregnant by someone in an opposing clique, I was perceived as damaged goods and no longer worthy of being a member, so I was jumped out. Meaning, I was beaten senselessly for 13 minutes, by both male and female gang members who I once considered my friends, my allies, my family. All I could do was curl up in a fetal position on the pavement of a dark alley to try to protect my womb, my baby girl.

The birth of my daughter sprung me into action. I owed her and the only way to pay the debt was to actually provide her a good life. I vowed to finish high school and get a good job so she would be immune to my destiny. But that was easier said than done. I found myself navigating through adolescent life, school, work, and parenting, alone. The pressures eventually just became too much and shortly after giving birth I dropped out of high school. I worked several mediocre minimum wage jobs to make ends meet with no avail; I eventually became a homeless teenage mother living off the kindness of others, food banks, and homeless shelters.

Then my life took a turn for the better, I was granted government assistance which included a monthly cash stipend, food stamps/vouchers, and health care for my daughter and me. As a result, for the first time in my life I had a sense of security and independence. I remember feeling happy because I was providing for my daughter. I was living large on the $400 per month – an enormous amount of money, more than I had ever seen. My world came crashing down once the government gave me an ultimatum to either go to school or lose my benefits. I remember feeling anxious, desperate, and hopeless, afraid for my daughter's future. So, under the threat I apprehensively went back to school.

I enrolled in the local community college to complete my general education diploma (G.E.D.). I had no idea what I was doing but I knew I had to be there for my daughter. After I completed my G.E.D. I stayed enrolled. To be honest it was mostly out of fear of losing my benefits. Surprisingly, this is where the trajectory of my entire family's life changed. A professor from the community college, offering his guidance, said that I should consider a career in education and suggested that I attend a talk about mathematics education being given by a passionate educator named Dr. Viji K. Sundar.

I remember walking into the room feeling like I did not belong. I was sure I was going to be found out. They—everyone in the room—would be able to read me and discover my past and label me as unworthy or tainted. The fear overcame me so much so that the audience transformed before my eyes into a mob. I remember wanting to curl up in a fetal position, in the back of the room in order to protect my being. Then she began talking. She was so energetic and passionate about mathematics education, women in education, women in science, technology, engineering, and mathematics fields, and serving marginalized communities that I just became enthralled.

On that day, from just that 45-minute talk, I learned to forgive myself. I learned that I could be a mathematics teacher regardless of my past experiences, gender, race, and socio-economic background. But most importantly, I learned that I could make a difference in the world. So, I committed to becoming an educator. Specifically, I committed to becoming a female, minority mathematics teacher. I remember telling myself that if I ever got to the point where I could stand in her shadow, I would know that I had made it. From that day forward, Dr. Viji K. Sundar became my beacon of hope. I now proudly self-identify and claim that I have reached that shadow and stand up in it, as a result of Dr. Viji K. Sundar's reach.

Inch by inch, math is a cinch. Yard by yard it is very hard. **(Viji Sundar)**

Angelo Segalla

Viji Sundar's love of life, family, students, colleagues, mathematics, and mathematics education will work its magic for some time. The memorial marker for Viji is not physical, it is the vast swath of honest teaching of a subject whose beauty is too often hidden behind pedagogical masks with fixed smiles; too many students are rightfully apprehensive.

As in the quote above, Viji pushed for curriculum and teaching to be upfront and honest, mathematically and pedagogically honest, with straightforward (human) help extended to students who needed help. She went for the heart, not the quasi-learning offered by computer-assisted tutoring and instruction in procedures.

This journal is one of Viji's ideas. I could not (she would not let me!) say "no" to being a co-editor. I relented but was worried about the extra work, of course. The work did not faze Viji amidst her busy schedule, nor did the fact that the title was inappropriate (since, at the time she asked me, the journal had little to do with the CMP). Viji's solution to this concern? We attended a meeting of the CMP and she floated the idea and name of the journal to the attendees. "Any opposition?" (I was ready for an onslaught!) None! Rather "It's a good idea, Viji." Whew!

> My admiration for Viji is huge. She was a mathematician and mathematics educator whose contributions advanced the science of mathematics education. As Viji's friend Dolores Arnett put it: Viji was like an atom - tiny and full of energy. She was always busy and brimming with new ideas. Whenever she encountered an obstacle, she tried over, under, around, and possibly through other dimensions. Viji was very caring and compassionate as well. I will miss this good friend.

A Subset of Joy Named Viji

Shandy Hauk and Mark Davis

What we learned from and through Viji Sundar would fill a stadium. We haven't got that kind of space here. So, we'll hit some highlights. Among other things, we learned the following.

- **Impatience!** – don't put off until tomorrow what can be done today and, if it should have been done yesterday, there's no time like the present.
- **Give love** – first by saying, "Thank you," and then in any other way you can. Even if the love is not returned, there is sweetness in it because the intersection of love and giving resides in joy, or mathematically: $\{\textit{Viji Sundar}\} \subset \{\textit{love} \cap \textit{giving}\} \subset \{\textit{joy}\}$
- **Always reach** – to extend a helping hand, to ask for assistance, to embrace life.
- Kindness is important and **Celebration** is even more valuable – celebrate now, don't put it off until tomorrow (see above, Impatience!).

Now called the *Journal of the California Mathematics Project*, this venue for communication founded by Viji Sundar embodies all of these principles. The editorial policy is to work with authors in sharing practices, decisions, research, and ideas in support of a better world through education inside and outside of California, with/in/through/from/for mathematics.

Journal of the California Mathematics Project
Volume **11**, 2022

Bringing Joy to Uninspired Teachers of Math Touchstone Strategies, Part 1†

Hal Melnick

Abstract. The first in a three-part series, this resource describes and illustrates two of eight Touchstone Strategies for teacher educators to use in their work with mathematics teachers. The article explores how to inspire teachers to find the joy in mathematics so they can support their students to do the same. Through a variety of tools, techniques, and helpful hints, the eight touchstone strategies in the series illustrate what high quality mathematics instruction looks like and how teachers can reframe their own thinking about mathematics to create deeper learning opportunities for their students. This piece, Part 1, introduces the collection and describes the first two touchstone strategies: *mathematical autobiography* and *"do" math*.

Introduction

Something surprising happened for me in the fall of 1973 during the very first days of my work as Assistant Director of Project Open Classroom, a progressive change project in Wayne and Pompton Lakes, New Jersey. I was hired to work with teachers in four school districts to assist with the districts' goals of changing teachers' practice to progressive, interdisciplinary teaching. Math was my focus. Midway through my very first set of workshops, I was shocked to find teams of teachers weeping. We were having fun, I thought, by using manipulative materials to engage deeply in conceptual math. Their words between the tears included, "How am I expected to teach this way if I never learned this way?" and "I've been teaching this all wrong for 10 years! What have I done to those kids?"

I learned pretty quickly that the emotions they felt were valid. Almost immediately I sensed the need to study this so that I could help them (and myself) deal with those emotions in order to have any chance of helping these teachers grow and change. I am forever grateful to those teachers and to one woman in particular named Ms. James. Ms. James repeatedly resisted change in her teaching. She both cried and argued with me. After two years, Ms. James was the teacher identified by the project as having the most changed classroom. It was her photo we put on the cover of our published book titled *Changeover*. It shows Ms. James sitting on the floor with two of her fourth-graders playing a concept teaching game about multiplication and factors using Cuisenaire rods. I reshaped my career research based on that work with Ms. James and all of her colleagues, and it helped me develop my mantra as a change agent: Embrace resistance.

† From an original report by Melnick (2018), this is the first of a three-part reprint with the permission of the author. The full report is available from the Bank Street College of Education website.

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In the past 43 years of teaching teachers about math instruction at Bank Street College of Education in New York City, I have maintained a clear commitment to two goals. First, I relentlessly insist that teachers understand conceptual underpinnings of big math ideas and, second, I hope to inspire teachers as mathematical inquirers. Throughout the years, I have encountered resistance in achieving both of these goals. Therefore, as a progressive educator, I have come to expect resistance to unfamiliar ideas. What I have learned through observation, research, and study is that, for the most part, the resistance is emotion-laden.

Disciplines like mathematics, science, anthropology, or even art in and of themselves, do not carry emotions, but learners surely conjure up feelings about learning them. For many well-researched reasons, feelings about math appear to be overwhelmingly negative, especially for North Americans (Battista, 1986; Bryant, 2009; Hembree, 1990).

Explanations for why elementary teachers could be uninspired teachers of math are varied. Myths exist that some people are hard-wired as math people and some are literary – writing – reading people (Boaler, 2017). Sadly, even though those myths have been debunked, I can still walk into any school today and surprisingly find some teachers who hold onto them. After years of careful personal study, I have developed my own theory as to why. I echo the words I first heard in the 1980s by Marilyn Burns, recipient of a Bank Street Honorary Doctorate and national math educator, that one learns math only by doing mathematics. She boldly suggests that you don't learn math when you simply learn about math. You learn math when you do math by actively solving real problems with friends: grappling with confusing ideas, making models to get yourselves out of confusion, asking each other questions, and arguing or debating relative solutions. After hearing Burns say this, I began to ask myself, "Might this be the source of the problem? Maybe people never actually did any mathematics themselves, ever — and maybe they never felt inspired?"

Emotion and Math

Anyone who carries the moniker "Bank Street Faculty" should be expected to address the social, intellectual, and emotional development of the learner, child, and adult alike. As faculty we refer to emotional development as the "affective realm." Research on affect and math education tends to define affect as having three subconcepts: beliefs, attitudes, and emotions. In their Psychology of Mathematics Education research summary Hannula and colleagues (2004) cite noted researcher Douglas B. McLeod, Professor Emeritus of Math Education at San Diego State, who offers distinctions between the four sub-concepts worth considering for my work with teachers. McLeod (1992) made distinctions among these and described emotions as the most intense and least stable, beliefs as the most stable and least intense, and attitudes as somewhere in between on both dimensions. Beliefs were seen as the most cognitive, and emotions as the least so. Later, DeBellis and Goldin (1997) added a fourth element, values. Most research on affect in mathematics education has addressed one or more of these four concepts. However, according to Hannula (2004), "the theoretical foundation beneath these concepts is not quite clear" (p. 106).

Multiple lines of research conclude that emotion is probably the most fundamental concept when we discuss affect. Researchers who have studied the psychology of emotions have used different approaches, and while there is no final agreement upon what emotions are, there is agreement on certain elements. Researchers seem to agree that, first, emotions are tied to personal goals. Second, emotions also involve physiological reactions that are distinct from non-emotional cognition. Third, emotions are also seen to be functional, they have an important role in human coping and adaptation (Goldin, 2000; Lazarus, 1991; Mandler, 1989; Power & Dalgleish, 1997).

Reflective Teaching

To become a successful Bank Street-educated teacher, one needs to be skilled as a reflective professional. We ask our graduate students first to reflect on what they have read and learned in academic courses and in conference group, a course in which instructors facilitate reflective conversations about the learning that happens during students' fieldwork. During fieldwork observations, advisors aim to help aspiring teachers "reflect-on-action" by discussing observed teaching moments. This "reflect-on-action" process continues throughout the year in advisement, with the hope that the new teacher will progress to "reflect-on-action" in future teaching when no advisor or coach is there to observe. The goal is for each learning teacher to become skillfully able to "reflect-for-action" in their planning and in their curriculum design (Schön, 2016).

This metacognitive process becomes better internalized for new teachers when personal emotions (the affective dimension) have been articulated, listened to, honored, and further reflected upon. One of my process goals is for new teachers to continually reflect, so that they can affirm their feelings as purposeful in the act of learning. According to Hannula and colleagues (2004)

> Consideration of meta-affect suggests that the most important affective goals in mathematics are not to eliminate frustration or to make all mathematical activity easy and fun. Rather they are to develop meta-affect where the feelings about emotions associated with impasse or difficulty are productive. (p. 113)

Many teachers I have taught indicate intensive reflections on their emotional engagement with learning during my course.

Teachers' Emotions and the Children They Teach

Recent research indicates that math-anxious teachers can have a negative impact on their first- and second-grade students' views about math and on resulting achievement (Beilock et al., 2010). Researchers at the University of Chicago found that math-anxious women teachers have a direct impact on the girls in their classes. They measured the degree of math anxiety of a team of female first- and second-grade teachers and the math achievement of these teachers' students (boys and girls). After one full year of being in any one of the math-anxious teacher's classrooms, it was more likely that girls (not boys, however) grew to: (1) endorse the belief that boys are good at math and (2) girls are good at reading. They also found that "Indeed, by the end of the school year, girls who endorsed this stereotype had significantly worse math achievement than girls who did not and than boys overall" (Beilock et al., 2010, p. 1860). Those findings suggest that in early elementary school, where teachers are almost all female, a teacher's math anxiety can have serious consequences for girls' math achievement. This could be a contributing factor to the high numbers of women who have avoided science, technology, engineering, and mathematics (STEM) study at US colleges. Although women fill close to half of all jobs in the US economy, they hold less than 25 percent of STEM jobs (Beede et al., 2011).

These conclusions bear out both in my observations as well as my naturalistic, qualitative data collecting. It is common to see only one or two men in my Math for Teachers courses at Bank Street during most semesters. Therefore, the large percentage of students attending Math for Teachers who, year after year, describe their feelings toward math as basically negative are mostly, but not all, female. I have always felt that their lack of inspiration about math will likely influence how they will teach math. My commitment stands to do what I can to help them reorganize their emotional relationship with math, so that they can be inspired and can inspire their students, be they boys or girls!

Suggested Instructional Practices for Teaching Math to Teachers

The suggested instructional practices I refer to here grow from my life of teaching at Bank Street College of Education, as first articulated in my dissertation research (Melnick, 1992). I aimed to study the nature of change that my students repeatedly told me they experienced while in my classes. I unearthed the themes of change they were experiencing before, during, and up to five years after completing Math for Teachers with me. Four themes were found: (1) grief in a graduate-level course, (2) healing, (3) reconstructing of one's math self-identity, and (4) unpacking their personal locus of change. Key strategies that emerged from that study and that I use in my instruction will be addressed in this work, which is intended as a resource for math teacher educators.

Throughout my career, in student course evaluations and conversations with fellow faculty, I have been described as an effective math teacher educator. I believe there are factors in my teaching that cause that. Most specifically, I believe it is the conscious effort I make to address the emotional component in my teaching that gives rise to these comments. I boldly address the affective realm in my instruction. In this work, I offer teacher educators a set of suggestions to consider when planning to teach groups of teachers who may present as having been taught by being told about math rather than by doing math. I will lay out eight touchstone strategies, behaviors, perspectives, or moves that I have enacted time and again as I taught teachers (see Table 1). Each is designed to help reveal the feelings students have about math and their perceptions of themselves as math thinkers.

Anyone teaching Math for Teachers at Bank Street or a similar course elsewhere may consider some or all of my indicated touchstone strategies. I use the term touchstone since I believe the term best characterizes strategies that include the affective dimension of teaching mathematics. I offer this truncated set of ideas in the spirit with which our annual Bank Street professional appraisals are conveyed: through consistent and self-revealing generative inquiry.

Table 1. Touchstone Strategies*

Strategy	**Brief Description**
1. Math Autobiography	Use a math autobiography as an in-class assignment.
2. "Do" Math	"Wow" students in the first class meeting by doing lab-type experiential tasks across Nursery School through Grade 6.
3. Collaborative Math	Model how collaborative group work is a special kind of group work.
4. Honoring Mistakes	Model how to honor mistakes and see them as opportunities rather than failings.
5. Journaling	Monitor everyone's learning through a math journal that is linked to class readings.
6. Work a Problem to Death	Work one intentionally perplexing problem to death, unearthing confusions that arise.
7. Non-Dominant Language	Teach one class meeting in a language other than the dominant one.
8. Concept Teaching Games	Have each student plan and share their own concept teaching game.

*The first two strategies are detailed in this article, Part 1, and others are the focus of Part 2 and Part 3.

Touchstone Strategy #1: Math Autobiography

Use a math autobiography as an in-class assignment in the first meeting of class.

Start with a reflective math autobiography in the very first meeting of class before you even introduce what the course is all about. Include in the math autobiography a simple request for just one word that captures each teacher's personal memory of learning math while in elementary school.

> *Please take a few quiet moments to reflect upon your memories of your mathematical life in your elementary schooling. What feelings did you develop during those years about yourself as a mathematically thinking person? What do you remember math lessons were like? What images are conjured up as you recall your teachers teaching you math? Include specific anecdotes if you recall them. Take as much time as you need to write this.* Follow up with: *Now, write just one word that captures your personal memory of learning math while in elementary school.*

Then, ask everyone to introduce themselves to the class by telling where and what age they teach and to share the one word they wrote. Write their words immediately on the whiteboard using three columns (see a representation of this exercise below), with strongly negative terms on the left, strongly positive terms on the right, and terms that carry little or no affect in the center. I ask them to talk to their neighbors about the responses. I push them to do some math and ask, "What percentage of the people in this class had less-than-positive feelings about their memory of learning math when they were in elementary school?" Those who didn't like math might not feel so alone. Those who liked math might begin to recognize a problem they didn't know exists: "This might happen to my kids, too!" In Table 2 is the chart compiled from one recent class, categorized as I might write on the board.

Table 2. Example chart with everyone's "one word" on the first day of class.

Negative	Words with Little Emotion	Positive
work	non-existent	pleasant
nightmare	memorization	Mr. Katz (neat)
pressure cooker	vague	confident
demeaning	unexceptional	tantalizing
useless		
dry		
conflicting		
yuck!		
blah		
murky		
threatening		

With these answers, we figure four-fifths or 80 percent of this class had less than positive memories. I tell them this is my challenge and my assistant teacher's challenge for the semester: How can we influence these feelings?

The word clouds in Figure 1 depict the attitudinal shifts in my courses before and after the eight humanistic, sensitive touchstone approaches are used to instruct teachers in constructivist progressive mathematics. The power of the language used in these anonymous one word feelings by the end in the last three semesters of classes I taught should convey why I believe that these eight strategies I have used are worth sharing.

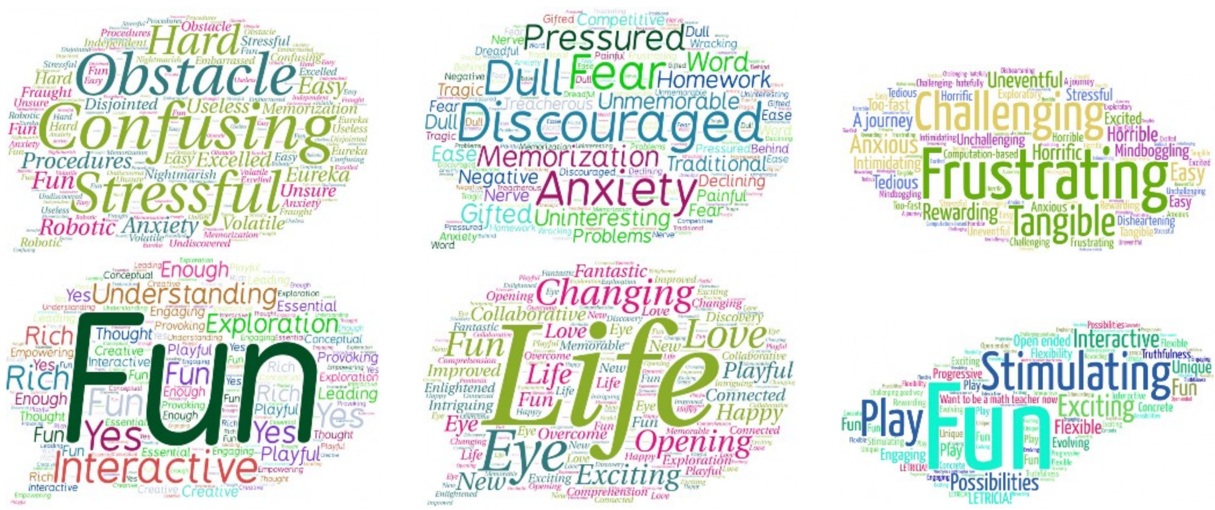
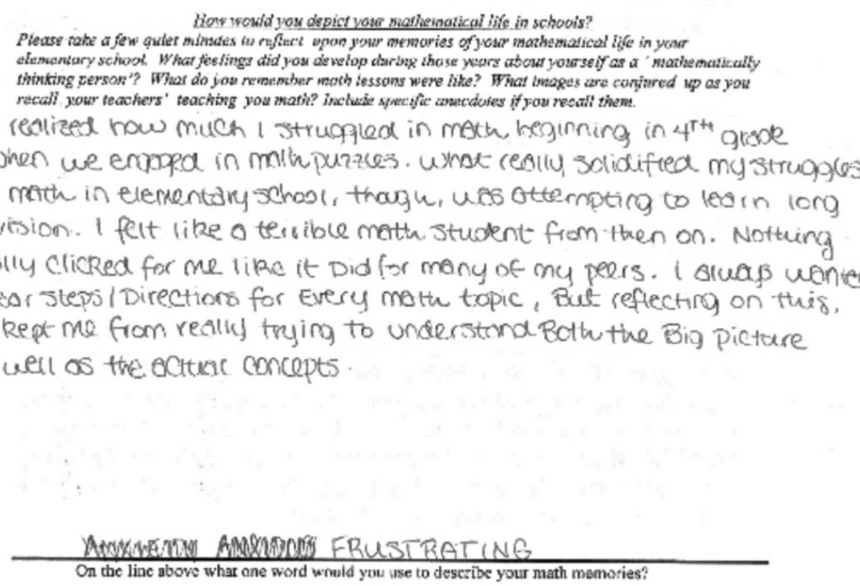


Figure 1. Word clouds for "one word" used before (top) and after (bottom) attending EDUC 540 at Bank Street. Each pair is from a different semester.

My teaching assistant and I make copies of everyone's math autobiography and read their one word and their further autobiographical comments with great interest (for a sample, see Figure 2). The first words they choose in Math for Teachers effectively help us tweak our plans for the balance of the course. We reference these words in many future activities. In private responses and individual responses to students, we often remind participants of the word they used on the first night. We ask them to continually compare their old and new feelings. I publicly admit to them that my not-so-hidden teaching agenda is that a reflective teacher will be a wiser teacher.

I have thought a lot about why these negative words keep on being selected by student after student. Based on data from math education classes I have taught for the past 43 years, prior to our first class at Bank Street, mathematics was done by only 20 percent of people attending Bank Street classes. The other


How would you depict your mathematical life in schools?
Please take a few quiet minutes to reflect upon your memories of your mathematical life in your elementary school. What feelings did you develop during those years about yourself as a ' mathematically thinking person'? What do you remember math lessons were like? What images are conjured up as you recall your teachers' teaching you math? Include specific anecdotes if you recall them.

I realized how much I struggled in math beginning in 4th grade when we engaged in math puzzles. What really solidified my struggles in math in elementary school, though, was attempting to learn long Division. I felt like a terrible math student from then on. Nothing really clicked for me like it did for many of my peers. I always wanted clear steps / Directions for every math topic, But reflecting on this, it kept me from really trying to understand Both the Big picture as well as the actual concepts.

FRUSTRATING

On the line above what one word would you use to describe your math memories?

Figure 2. Sample of the Math Autobiography (Emily, 2017).

80 percent typically learned about mathematics, often in a rote fashion, and were usually taught by teachers who diligently gave them rules, formulas, and routines to memorize. Those teachers then expected their students to regurgitate what they were told, not necessarily to consider whether or not they understood any of those rules or formulas.

In addition, these students tell us that they learned not only to study for a math test and to get right answers by memorizing rules and formulas, but also not to expect to understand the reasons why any of it works. Math educators today know that a child should never be asked to memorize something he or she does not understand. The standards documents by the National Council of Teachers of Mathematics (NCTM, 1989, 1991, and 2000) and the Common Core State Standards (2010) emphasize teaching math for understanding. If math is taught simply to memorize, what may result for learners is a kind of math thinking paralysis, which is useless for building upon in later years (Boaler, 2016).

These short math autobiographies begin to signal to students my request for their feelings. I let them know I am just as interested in how they feel as in what they think. This continues in their journals, where they write about their course readings, their class experiences, and their feelings about our class. The feelings depicted in the "before" word cloud visuals are generally the result of being taught day after day with a rote approach. Since there are so many variations to the rote teaching approach in the literature (Mayer, 2002; Munter et al., 2015), I have come to call it the "teaching by telling" method. Learners in teaching by telling math classes never really had to do much other than memorize predetermined rules, formulas, and procedures that other people once discovered when they were doing math.

Here is a caveat to my above premise. Some people did just fine in math, and they were taught math with the traditional teaching by telling method. How is it that some of these adult students made it through and even went on to successfully study higher-level math and appear to now fully understand the roots of math concepts? These learners exist (although generally, I find they are only about 5 percent of those I have taught). They possess an admirable capacity to make connections between procedures and meaning on their own. Some people can connect memorized, separated facts and details, linking those disparate facts to underlying root concepts. For example, they might know that division is directly related to multiplication; in fact, the undoing of multiplication. They early on noticed that if you learned your multiplication facts, you automatically knew all your division facts. And they intuitively saw that long division was simply one procedural strategy to unearth the dimensions of a rectangular area whose area is called the dividend. They see divisor, dividend, and quotient as three numbers that are related to each other, and that all are open to multiple strategies for finding and visualizing all three as parts of a rectangle. They see fractions as division because it is! They can then use this information flexibly, like engineers do in design work or as others do in fields requiring any kind of relational thought. Math is seen by them to be inherently normal thinking. Additionally, I often find out from these students that some of that 5 percent group were taught by teachers who ran math classes that allowed kids to learn by doing, not by being told. With these progressive teachers, even years ago, students explored, conjectured, hypothesized, tested hunches, worked collaboratively, and discovered meanings in their math classes.

But what about the other 95 percent of my students? Many were unintentionally rendered disabled in math because they were not allowed to do math. They merely learned about math. The feelings they were left with are clearly depicted by the obviously negative words in the "before" word clouds. In the Math for Teachers class, the math autobiography writing on the first night establishes a firm reflective stance for the entire course. The impact of those one words on learners is described by these quotes from the student summation journals I ask my students to keep as part of the course.

On completing the math autobiography in the first class:

> Even just writing that mini-math autobiography from the outset felt like a mini-math demon exorcism, and starting on a clean slate. Not quite the word count real estate to nail my 95 theses to the Church of Math Door, but it felt a bit cathartic. I was also fascinated by the words people were asked to provide in association with our math

> experiences. I believe my word was "stress," and I don't recall any of the other words having a positive connotation. I take zero pleasure from anybody else's struggle with math, but, at the risk of sounding trite, it made me feel a little less alone in my self-imposed math exile. I know that I will have to recalibrate my entire outlook and approach to the subject, which will not happen overnight and will be an ongoing process throughout my career as a teacher, but I'm ready to go there. (Caitlin, Spring 2017, start of class)

At semester's end, I ask that you again listen to Caitlin's feelings expressed. Consider her evolution. Here she reflects upon the word she wrote in her math autobiography at the start of the course. She writes about her evolution as a learner in our Math for Teachers class. She sums up what happened to her over the semester:

> For the life of me I can't find my original math autobiography, but I luckily detailed what I had written in my first journal for this class. The word I picked when asked to sum up my experience with math was "stress." While I had blocked out a lot of my painful math experiences–many of which I've recovered from through the process of autopsying my own math education in our journals–I very clearly remember my last day of math in college and feeling a sense of elation. Pure joy, really. I was done with math and would never have to deal with math again. Flash-forward a decade and I'm undergoing a career change going from the magazine world to the education world and, surprise! Apparently math wasn't done with me. I dreaded our first math class, not knowing what to expect. But, at the risk of sounding completely cheesy, I truly feel like a different person with the subject four months later. Mainly, that I no longer fear the subject the way I did. I realize that my relationship with the subject will continue to evolve and develop through the years, more so when I actually start teaching, but I feel completely empowered. I'm actually shocked by how much, truly. I can't press the reset button for my own trials and tribulations with math, but as a teacher I now feel that I have the tools and resources to prevent my future students from experiencing what I did. (Caitlin, Spring 2017, end of class)

I firmly hold that the experience of writing a reflective math autobiography on the first night of class provides a firm starting point for growth through reflection.

Touchstone Strategy #2: "Do" Math

"Wow" students on the first night of class by doing lab-type experiential tasks for nursery school through grade 6.

With little fanfare (but a lot of preparation on my part before students arrive) I intentionally aim to "wow" my students right away. My goal is to shock them with a colorful and joyful view of what math teaching can look like across grades N-6. Graphs with materials to manipulate, and at times even taste, are up on walls or on tables for different stages of abstraction. A classic racing dice game is there to be played in Kindergarten or in more depth at Grade Six. Charts exist on walls with ponderable math questions, materials are on the table for measuring and drawing conclusions from those measurements about the idea of the mean, etc. These tasks surround the room at lab tables and on walls (Figure 3 has photos of six examples).

As soon as they fill out their math autobiography, they get up and do the 11 or 12 tasks around the room. I always tell them to find a friend and work cooperatively on the tasks for support.

They wander with smiles, happily tasting white or purple grape juice and place their empty cup on one of two columns on the vinyl coated graph. The resulting lines of cups demonstrate a three-dimensional "real" graph showing drink preferences for 4- to 5-year-olds. A noticeable active noise erupts and laughter ensues. They are immediately "doing" math. At one learning station, they are expected to measure their heights with little white Cuisenaire Rods, graph the data using post-it notes, and ponder the shape of the emerging graph and where the next person will most likely place their post-it. At another station they have to toss two dice as many times as they can, recording on a 2-12 chart when each sum turns up. Which sum do they anticipate will win? Why? A question on their worksheet asks which sum they think will reach the top of the grid first. They enter their results and we compile a Sums that Won Class Chart. What is the theoretical probability of your "winning number" winning the next time you play? This and about eleven other tasks across the age span generate a lot of open conversation, laughter, and pondering about what the course might entail.

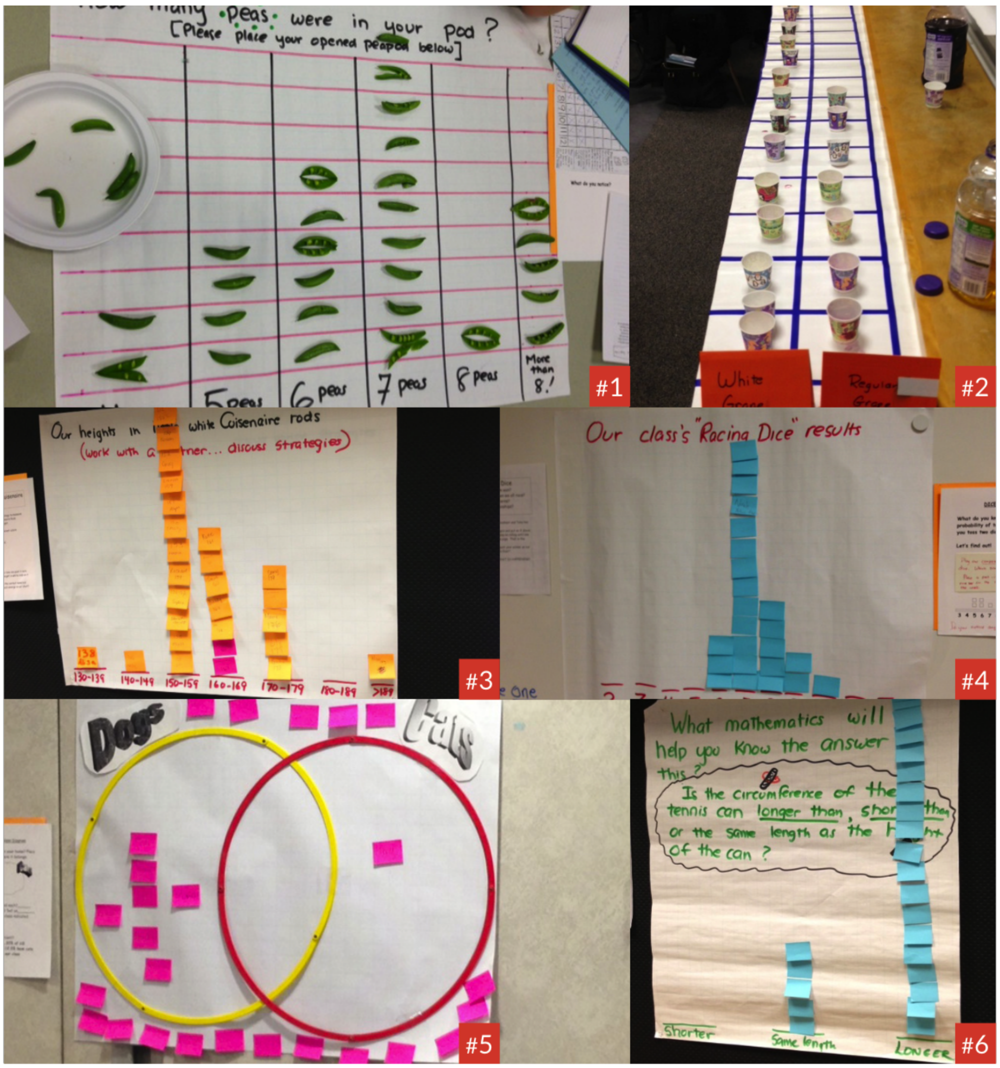


Figure 3. Six examples of tasks done by students in the first class meeting.

#1 How many peas are there in your pea pod?
#2 Our preferred juice drink.
#3 Our class's heights in white Cuisenaire Rods.
#4 Our class's Racing Dice results.
#5 Do you have a dog, a cat, or neither at home?
#6 Is the circumference of the tennis can longer than, shorter than, or the same length as the tennis can itself? What math that you likely learned in middle school will help you answer?

Things they write inform me about their reaction to that night and invariably begin to inform my assistant and me about their prior math lives. Right away we start hearing expressed emotions, fears, excitement, and expressions of attitudes and values. The affective realm is out there in full view. The seeds for the course are planted.

Here are a few of their comments evidenced in their math journals after the very first night of class:

> Doing the exercises, from the racing dice to the pea pod estimations, was really compelling. Something that might seem simple on the outset actually stimulates much deeper thinking, even in adults. Why did most people end up with sevens and eights as a total with the dice? Why did people choose the white grape juice over the red? These group activities made me realize that I will have to be in a constant state of questioning and will have to likely break up any preconceived notions and assumptions I've had in the past, in order to get the most out of this course (Caitlin, Spring 2017).
>
> Last week we had our first class of Educ 540. Walking out of class that night, I felt something I had never felt before; excited about math. To me, this was a big deal. The reason being that in the past, whenever walking in or out of a classroom where the topic involved math, I would have been overcome with anxiety.
>
> This class began with our professor asking us to recall our experience with math before grade six and one overall word to describe this experience. I wrote down some of the phrases my classmates shared; stressed, confusion, anxiety, trauma, and horror. Most of my classmates shared a word that implied their experience with math before grade six was negative. I began to analyze why exactly my experience was so negative, and I realized that learning math lowered my self-esteem and confidence. I was unable to grasp the concepts fully we were learning in math class, and this caused me to classify myself as "not smart." This feeling followed me each year, and before I even entered into a new math class, I already had that feeling of inevitable failure.
>
> However, this class made me excited about math because I want to learn how to best teach math in a way that students will not feel the same way myself and my classmates felt. There are so many fantastic resources and ideas about teaching math in a way that fully supports students, and I can't wait to learn about them. My professor also made it clear that he believes each student learns best in a different way. This philosophy made me hopeful as I wholeheartedly agree and believe in the importance of considering each student individually. When I was younger, I wished my teachers would have considered the fact that each student learns differently and just because one strategy worked for my classmate, that didn't necessarily mean it was appropriate for me (Katherine, Spring 2017).

I am always amused by the delight on people's faces during this first session. I observe and take note of each person's reaction. Is there a sign of joy; of fear, of confusion? Does anyone appear very uncomfortable? Towards the end of that first class I remind people of the electronic link that was sent to them a month earlier in my introductory welcome letter. The emailed letter had included the course outline, lists of assignments, and required books for the course, however I never spend the first night with such details. I ask them to carefully read those materials again and come with questions, all of which will be addressed next session. My intention is to be invitational and inspirational on the first night. I will admit that occasionally some people drop the class as a result. Sometimes they are just not ready to handle the kind of shift this night lays a foundation for. When they are ready they will return.

Acknowledgements

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Using Pitfalls to Support Middle School Mathematical Discussion and Equity

Kathleen D'Silva, Elizabeth Dyer, and Jodi L. Davenport

Abstract. This study examined how three seventh-grade teachers implemented discussions of mathematical stumbles and errors (pitfalls) while using an instructional supplement. Using a "Math-Talk" framework as a lens, results indicate that some teachers faced a conflict between what they saw as important in maintaining student trust (e.g., validation of correct answers) and giving time and attention to pitfalls. One teacher who celebrated mistakes as learning opportunities, saw that discussion of pitfalls lead to more equitable student engagement. We examined the variation in facilitating discussion across teachers and offer a possible extension to the Math-Talk framework.

Theoretical Perspective and Background

Access and equity in mathematics classrooms require that all students "participate meaningfully in learning mathematics" (National Council of Teachers of Mathematics [NCTM], 2014). Many equitable teaching methods, such as focusing on mathematical reasoning and mathematical practices, involve the use of language (NCTM, 2014; Moschkovich, 2013). In an equitable classroom, all students participate in discourse. A mathematical discourse community is generated when the teacher and students agree to work on meaningful communication about mathematics (Willey et al., 2017).

Mathematical discourse "includes the purposeful exchange of ideas through classroom discussion, as well as through other forms of verbal, visual, and written communication" (NCTM, 2014). Mathematical discourse communities improve student problem solving and deepen conceptual learning (Hufferd-Ackles et al., 2004; Murata et al., 2017) while promoting equity (NCTM, 2014; Michaels et al., 2008). Walshaw and Anthony (2008) said "explanations stimulate, challenge, and extend other students' thinking" (p. 25), but caution that discussions only enrich classwork when all students are included.

Seemingly productive discussions can leave out some students or may not yield deep, conceptual learning (Murata et al., 2017; Walshaw & Anthony, 2008), thus it is important that all students participate in the discourse in meaningful ways. But how is that possible when students are not always "correct?" Focusing discussion on explaining wholly or partially incorrect problem-solving strategies, or pitfalls, may be a key strategy in creating effective conversations because it communicates that all ideas are valuable (Booth et al., 2013).

Barbieri and Booth (2016) found that when lower-performing students reflected on and explained incorrect solutions, or pitfalls, their algebra learning improved more than comparable students

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who only examined correct solutions. When questions focus students' attention on common pitfalls, students must directly confront common misconceptions and are less likely to make similar errors in the future. Explanations related to pitfalls, can generate inclusive, productive mathematical discussion. When conversations acknowledge, confront, and unpack pitfalls, they support student sense-making (Booth et al., 2013). The current study explores how a classroom focused on sense-making around mathematical pitfalls might yield more effective and inclusive discussions.

The Math-Talk Learning Community Framework (Hufferd-Ackles et al., 2004) describes the developmental trajectory needed to create "math talk learning communities" where the primary goal is to "understand and extend one's own thinking as well as the thinking of others in the classroom" (p. 2). The framework emerged from a case study of one teacher who, over the course of a year, moved from a traditional pedagogy to using more reform-based practices for success in whole class discourse. The framework was then refined based on data from three other classrooms and has been used in numerous studies since its development (e.g., Murata et al., 2017).

The base level, Level 0, of the Math-Talk framework describes a traditional teacher-directed classroom while at the top level, Level 3, students have ownership of the classroom talk and the teacher is facilitator, co-constructing knowledge and discourse with students. Within the Math-Talk framework are four components: questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. In particular, the *source of mathematical ideas* attends to who owns mathematical ideas, including who decides that an idea is mathematically valid.

See Table 1 (next page) for brief descriptions of each math-talk level for *source of mathematical ideas*. In the current study, we used the construct of *source of mathematical ideas* as an interpretive lens to explore the ways teachers viewed and used mathematical pitfalls for classroom discussion.

Methods

Research Questions. This paper focuses on two exploratory research questions: How do teachers make sense of the role student pitfalls play in class discussion? How might the handling of student pitfalls support/hinder equitable access to mathematics?

Approach. We used qualitative methods based on constant comparative coding of interview and observation data (Patton, 2015), first for themes, then for patterns, and finally for relationships and distinctions among patterns using the Math-Talk framework. Member checks were done by sending participants drafts of this manuscript and requesting feedback on the accuracy of the analysis.

Participants. All three case study teachers, Rita, Sean, and Jane (pseudonyms) taught seventh grade math at public middle schools in California. All had secondary mathematics credentials, at least six years teaching experience, and had taught at their schools for at least three years. Participating teachers engaged in a pre-implementation workshop with virtual follow-up sessions. Subsequent case study data collection was March to May 2018.

Table 1. Math-Talk Framework: Source of Mathematical Ideas (Hufferd-Ackles et al., 2004, pp.88-90).

Level 0	Level 1	Level 2	Level 3
Teacher is physically at the board, usually chalk in hand, telling and showing students how to do math.	*Teacher is still the main source of ideas, though she elicits some student ideas. Teacher does some probing to access student ideas.*	*Teacher follows up on explanations and builds on them by asking students to compare and contrast them. Teacher is comfortable using student errors as opportunities for learning.*	*Teacher allows for interruptions from students during her explanations; she lets students explain and "own" new strategies.* Teacher uses student ideas and methods as the basis for lessons or mini-extensions.*
Students respond to math presented by the teacher. They do not offer their own math ideas.	Some student ideas are raised in discussions, but are not explored.	Students exhibit confidence about their ideas and share their own thinking and strategies even if they are different from others. Student ideas sometimes guide the direction of the math lesson.	Students interject their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare and contrast and build on ideas. Student ideas form part of the content of many math lessons.

**Teacher is still engaged and deciding what is important to continue exploring.*

Setting. Teachers taught several two-day lessons from a supplemental curriculum focused on exploring student thinking, *Math Pathways and Pitfalls*® (MPP; Moorjani & Kao, 2019), as part of a larger quasi-experimental study. Lessons started with a class discussion of the lesson purpose and relevant "math words" (i.e., vocabulary), followed by a starter problem, meant for students to work on independently. Students then considered and discussed the printed work of two student characters: one correct and the other with a common pitfall. MPP is based on the principle that jointly examining correct work and confronting pitfalls supports mathematical discourse community development. During instruction, this is realized through reflection and exploration of the problem solving of the characters in the text and of students in the room. MPP provides teacher prompts and a poster of "discussion builders," or sentence stems to scaffold students' classroom conversation (e.g., "I have a question about [another student]'s idea…").

Data Sources for the Research. Data collected included an initial interview, observations of two different two-day MPP lessons, and, after each lesson, an interview with the teacher. In the initial interview, we asked teachers their perspectives on how discussion related to equitable access to the mathematics (Part A) and strategies for building mathematical discussions (Part B). Observation data were collected as a running record with prompts to focus observers on equity and access with attention to both teacher and students. The first post-observation interview was similar to the initial interview but related the ideas to the observed lesson. The second post-observation interview focused on curricular tools that helped or hindered discussions as well as teaching dilemmas associated with discussions.

Results

In their interviews, teachers talked about the benefits and challenges of using pitfalls in classroom discussions. These reports and observations on discussion within the Math-Talk framework are given below.

Sean - Interview. Sean said students learned from the pitfalls in the curriculum, "knowing the answer is right or wrong, the kids have an easier time explaining why it is correct or explaining and checking why it is wrong." However, he said students were hesitant to share and learn from their mistakes. Sean said he did not ask students to present incorrect solutions if he had checked their work because he did not want to violate their trust. In the interview, Sean said the "beautifully-done MPP days, is when I talk the least. So, if yesterday I talked 55% of the time and today I talked 45% of the time, that is a successful day." He thought the MPP curriculum's focus on pitfalls supported student engagement. Also, Sean felt he encouraged engagement by giving students points for participating in discussions but limiting the maximum points they could earn to make space for less eager students to contribute. Engagement was further strengthened, he said, by giving students time to share in small groups before sharing with the full class. Sean was very aware of the percentage of students who spoke who were of different backgrounds (e.g., race, gender, and math ability) and academic habits. He was very focused on ensuring diversity of voice and found that the curricular focus on pitfalls, combined with his strategies increased engagement. However, he noted that students who did not normally participate (on non-MPP days) did not participate in the MPP lessons either. He desired more inclusive engagement.

Sean - Observation. In Sean's MPP lessons, the math-talk community demonstrated some Level 2 elements, with students sourcing some mathematical ideas. Sean followed the general format of MPP lessons, asking students to discuss worked examples and pitfalls in pairs, followed by students presenting their work to the class. He elevated the status of student voice by calling presenting students "maestro/a" and "professor," and facilitated students asking questions of each other. He asked students to respond to the presenter using the discussion builder sentence stems. Sean's prompts probed student thinking and he occasionally asked students to make connections between different problem-solving strategies. The community also demonstrated Level 0 and Level 1 elements, with Sean becoming the source of ideas when students struggled. When Sean probed students' thinking, students often did not respond, and Sean narrowed his questioning towards more short answer/recall prompts. If students did not volunteer to present or struggled, Sean explained, asking for student input. While Sean stated in class that student errors were valuable, errors were not used as a platform for learning. Neither teacher nor students probed thinking around errors made by either the fictional students or those in class. When students presented incorrect solutions, Sean or another student presented a correct solution immediately after. Sean was worried about violating student trust by asking a student to present work if he knew it was incorrect.

Jane - Interview. Like Sean, Jane said she felt that students were learning from worked example pitfalls, but that they had troubles learning from their own mistakes. She, too, felt that trust was an important aspect of the mathematical discourse in her class. Jane felt she would betray trust by asking students to present incorrect responses, "Yeah, I would feel really bad about seeing that a student had something wrong and saying, 'hey, go up and show the class.'" She also noted that pitfall discussions increased engagement, catching the "in-between kiddos," though she did not think any curriculum could engage students struggling in school in general. Unique to Jane was her focus on what individual students did or did not understand about particular mathematical problems. For example, she was surprised when a student, discussing a worked example pitfall, made clear he did not understand what x represented, "he was saying 'how can you add a number, 15 to a number, if you don't know what that is?' And I was like 'huh? ... they're at a level where they should understand that."

Jane - Observation. Like Sean, Jane's class included some Level 1 and Level 2 sourcing of mathematical ideas. Students discussed in groups and then presented in class. She probed student thinking using prompts from the curriculum, and used wait time and asked "Anyone have anything to add?" to encourage engagement. She suggested people try, saying that mistakes are how people learn. These Level 2 attributes were mitigated by students' apparent (to Jane) lack of confidence in their ideas. It was sometimes hard to find volunteers and students often did not know how to respond to prompts. At these times, Jane would have students work more in groups while Jane coached an individual. The individual attention would continue until Jane and the student agreed the student was ready for presenting; meanwhile, other students became disengaged or struggled unproductively. Similar to Sean's classroom, if incorrect work from a fictional student in the materials or a student in the room was presented, someone quickly gave a correct solution without interrogating the sense-making behind the pitfall.

Rita – Interview. For Rita, pitfalls were something to be celebrated. Airing student mistakes and the shared value of pitfalls for learning lead to animated full class discussions. Students who thought they understood were pushed to learn more because confused students asked them questions. She said, "We clap if you do it wrong because you get it out of the way in classwork and homework and then you don't make that mistake on tests." The trust established within the class group appeared to support such celebration. According to Rita, when outsiders came to the class, students were less likely to share, "they didn't want to say anything wrong, so they were very careful and spoke a lot less than a typical regular lesson." Student engagement increased for many but not inclusively, for all students, in the experiences reported by Sean and Jane. Whereas for Rita, there was more equitable engagement, which she attributed, in part, to valuing pitfalls:

> I honestly believe that if you didn't know who my resource kids were and my other kids, you couldn't pick them out [...] In the past, you could have done that all year long. These kids now feel they have value in what they say, [...] so that has brought just a wonderful culture.

Rita – Observation. Rita was observed for only one typical class period, not the full two two-day MPP lessons required for the study. Considering this limitation, the data support Rita's descriptions of her class in which Level 3 elements were visible (even though Rita reported students were more reluctant to share due to the observer in the room). Rita put student thinking and examining misconceptions forefront: she shared with the class all ideas she heard discussed in group work and supported students' group and full class discussions. All students appeared actively engaged in small group discussions. In one instance she used a student misconception as an opportunity for further discussion. A couple of students shared the hypothesis that "when you multiply a number by a fraction, the product will be smaller." Rita had students discuss this in small groups during which time there was informal sharing between groups. In full class, a student shared "if you multiply by a fraction that is greater than 1 the product is going to be greater." One of the original students responded, "I changed my mind after looking at more examples." The observations provided evidence of the culture Rita described where students were willing to present novice understandings and all students engaged mathematically in discussions.

Discussion and Conclusions

Tension: Trust vs. Pitfalls. Sean, Jane, and Rita all mentioned that their students learn by making and thinking through mistakes. Sean and Jane were concerned with making student pitfalls public. Jane said, "When they go up to the front, they want to be right, because it is scary for them, whether or not you tell them a thousand times 'pitfalls are ok, it is part of learning.'" She thought students believed that only the correct answer is valuable and that she could not "undo" that in a few lessons. The observed full class discussions were centered around correct solutions or correcting a fictitious students' mistake, rather than centered on understanding student thinking around errors. This may have compounded student reluctance to share incorrect work. Sean and Jane seemed to struggle with the tension between valuing pitfalls for learning and upholding student trust and safety. In contrast, Rita reported students would not hesitate to share incorrect solutions or ask for help and the observation provided evidence of students' comfort sharing preliminary ideas without knowing about correctness. Making sense of pitfalls in class discussions and celebrating pitfalls seemed to be key differences between Rita's math-talk community and those of Sean and Jane and may be a key element of Level 2 sourcing of mathematical ideas (i.e., "using student errors as opportunities for learning"; Hufferd-Ackles et al., 2004, p. 89). For Rita, student trust was not about saving students from embarrassment around partially or wholly incorrect thinking, but rather about trusting that student ideas, whether pitfalls or not, were valuable for learning mathematics.

Source of Ideas: Teacher Goals and Student Actions. Sean wanted discussions focused solely on student ideas, with students questioning each other and the teacher keeping the conversation focused. On the surface this aligns with Level 3 sourcing of mathematical ideas. Similarly, Jane wanted student thinking and ideas to be foregrounded, but with the teacher taking a more prominent facilitation role, as described in Level 2. However, they both struggled when students did not volunteer to present or comment on each other's work. The students may have been expecting a Level 0 or 1 classroom while the teachers were aiming for Level 2 or 3. Authors of the Math-Talk framework described each attribute as being "developmental trajectories in teacher actions and students' actions [that] were derived from the data" (Hufferd-Ackles et al., 2004, p.87). Students and teachers likely must progress through each level in order, which may explain teachers' struggles to take their math-talk communities to the next level. Sean and Jane may have bypassed some of the transitional steps.

Pitfall-Focus May Create More Equitable Engagement. By focusing on student thinking, with particular attention to student reasoning around their own pitfalls, Rita believed she reached more equitable engagement and outcomes for students. Sean and Jane did not use student pitfalls as opportunities for learning, which may be related to their struggles to engage a wider variety of students. By focusing discussion on correct strategies, Sean and Jane were acting as arbitrators of what ideas were and were not valuable in the classroom, leading to less engagement. Jane grappled with this, "The idea of going up, they are going to want to work harder to make sure they are more accurate. But that's hard because you get kids who avoid doing the work because they don't want to [share work]." Spending more time validating, interrogating, and making sense of student thinking around pitfalls may lead to more and different students sharing.

Study Limitations. Our study conclusions were constrained by the completeness and inherent variability of our teacher self-report data: Sean did not complete part of the first interview due to time limitations, and though Rita participated in all interviews, we were only able to observe one typical day of one two-day MPP lesson. Finally, many factors that affect the nature of classroom discourse, such as culture, language, race, and ethnicity (Moschkovich, 2007), were not investigated in the current study. As this was an exploratory study focused on discussion, not just pitfalls, teachers were not systematically asked about the complete set of themes that emerged relating to pitfalls, and we save this investigation for future research.

Educational Importance of the Research

Focus on Pitfalls. This exploratory study indicates that celebrating and focusing on reasoning behind student pitfalls may support quality student engagement and higher levels of math-talk. When pitfalls are discussed publicly and related to correct solutions, students may find sharing their own ideas and pitfalls to be valuable. However, teachers and students may be uncomfortable discussing student pitfalls and, thus, focus on correct solutions. Research should explore these dynamics in relation to the socio-cultural contexts they are embedded in.

Math-Talk Framework for Upper Grades and Equity. Sean and Jane's challenge to engage all students in discourse while maintaining trust is not explicitly addressed in the Math-Talk framework. Students in seventh grade may be more likely to worry about how they are perceived by peers, especially when making pitfalls, than the early grade students in the Hufferd-Ackles et al. (2004) work. Middle and high school teachers may benefit from tools that explicitly address how to build student trust in a math-talk community, with particular attention to maintaining trust while discussing what teachers perceive as sensitive topics such as students' pitfalls.

While the Math-Talk framework gave a lens for understanding the discourse in Sean, Jane, and Rita's classroom as a whole, it does not address individual student engagement. Based on these findings, class level engagement builds on individual engagement and is an important dimension of classroom discourse. For example, not all students engaged in quality math-talk in Sean and Jane's classrooms. The research community, and likely teachers, could benefit from a mathematics classroom discourse framework that could pick up on differences in student engagement within a classroom.

Supporting Math-Talk Community Development. This study points to some areas of focus for teacher educators and curriculum developers in helping support teachers in developing a math-talk learning community. First, math-talk communities develop over time, likely going through each of the developmental Math-Talk framework levels. Perhaps teachers should not be encouraged, as in this study, to engage students in Math-Talk framework Levels 2 and 3, without intentionally moving students first through Levels 0 and 1.

Second, teacher educators and curriculum developers can help teachers and students become more skillful in focusing discussion on the reasoning behind pitfalls. In particular, students may learn the content more deeply through interrogating pitfalls as the teacher supports a community norm in which all students' ideas are valuable for learning.

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A Curriculum Review Checklist to Support Teaching for Robust Understanding

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Abstract. Part of the professional work of teaching includes selecting curricular materials. The selection might be small (for a lesson or a project), medium (for a course one is teaching), or large (for a school or district). Sometimes the selection is done by one person, sometimes a committee. Whatever the circumstances, research suggests that having a framework for examining materials and methods is valuable. The first half of this report describes how such a framework might be designed in alignment with the goals of the Teaching for Robust Understanding approach while also supporting responsive instruction. The second half of the report provides an example, in checklist format, that the reader might use immediately. The checklist consolidates the research-based experience of veteran teachers into a useful tool for a newer teacher.

Introduction

As teachers we apply a personal "theoretical framework" about what makes for good teaching in the decisions we make about instruction. But our framework may not be well-organized or even thoroughly examined. The checklists in this document, which can be used to help a teacher decide on the usefulness of a curriculum, are based on a framework we (the authors) have found useful called Teaching for Robust Understanding (TRU; Schoenfeld et al., 2016). The checklists offer a structure for considering all five of the dimensions defined in the TRU framework (see Figure 1). The checklist is a tool for ensuring that the content is appropriate, the cognitive demand is rigorous and supportive, that all students can access the learning opportunities, that students get to have a say in their learning and build their identity as learners, and that the teacher has direction for next steps after a given lesson. Using these checklists can help teachers select quality curricular pieces and improve implementation of existing curriculum; in addition, it can help to shape and clarify a teacher's own theoretical framework and provide a starting point for future professional learning. To use the checklists, it is important to first understand what the TRU framework dimensions are, what they look like in practice, and how they are related to improved student learning outcomes.

The checklists are also influenced by the principles of Complex Instruction (Cohen & Lotan, 2014). TRU and Complex Instruction both provide ways to unpack teaching, though in somewhat different ways. Complex Instruction focuses on the intersection of three key components: classroom norms and student roles, status interactions, and groupworthy tasks (Horn, 2012). Complex Instruction is primarily about how to establish and maintain positive relationships among and between all students, the teacher, and the mathematics itself, which is essential for equitable access to the curriculum. The TRU framework is more focused on what the

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outcomes should look like—that is, how to observe or evaluate the effectiveness of a teacher's practice. We believe TRU is also a useful tool for examining curriculum, particularly for identifying curricular supports for the kind of teaching promoted in Complex Instruction. Our intent in writing these checklists is to help you identify a curriculum strong enough, deep enough, and supportive enough to be a base for robust teaching and learning.

The Five Dimensions of Powerful Mathematics Classrooms				
The Mathematics	Cognitive Demand	Equitable Access to Content	Agency, Ownership, and Identity	Formative Assessment
The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.	*The extent to which students grapple with and make sense of central mathematical ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with tasks that are not low level but not impossible. The level of cognitive demand should be conducive to what has been called productive struggle.*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.*	*The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.*	*The extent to which classroom activities elicit student thinking and subsequent instruction responds to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.*

Figure 1. Summary of the TRU framework (Schoenfeld et al., 2016).

Position of the Authors

Before moving to the intentions around and uses of the checklist, we offer short professional biographies. These allow the reader insight into the people behind the checklist offered here.

Daniel. When I first started teaching Algebra, I tried incorporating problem-solving in addition to the required procedural work. But this was the beginning of NCLB, and each year, the STAR tests continued to grow in importance. My school was laser focused on STAR scores, so my teaching became more and more procedural and targeted to the released questions. Problem-solving and anything else that was interesting disappeared. Up to this point, I had been teaching with an emergency credential, but by my 5th year of teaching, I decided that I really did want to be a teacher. I enrolled at San Jose State University, and got my credential in the evenings, as I taught full-time on an intern credential (which means I never really had a mentor teacher). The program's overall quality was mixed, but I had a very good experience with my mathematics methods teacher, Ferdie Rivera. From him, I learned about the socially constructed nature of mathematics learning, and I learned a great deal about how to actually structure learning experiences for my students. My teaching began to improve again, even under the oppressive environment caused by NCLB. But by my 10th year of teaching, I felt like I had plateaued, and I was still not getting my students to learn or be engaged in the ways I believed they could. I was building strong relationships with them, but not helping them build strong relationships with math. I was starting to feel like my choice to teach mathematics had been a mistake, and that I could not keep doing it for much longer. The adoption of the Common Core turned things around for me, because it finally allowed us to teach thinking, communication, and problem-solving skills. The transition to Common Core, however, was a tough one for many teachers. And even now, 10 years later, many mathematics teachers are still teaching procedures and concepts, and not putting the mathematical practices at the center of learning. I eventually became the co-chair of the mathematics department at my school site, and began encouraging people to use new instructional practices that focused on student discourse and problem-solving (math talks, notice and wonder, MARS tasks, 5 Practices, Routines for Reasoning, modeling tasks, and so on). I read tons of books, blogs, and twitter feeds and really increased my own mathematics teaching knowledge. I am currently finishing up my masters in mathematics education at San Francisco State, where I've learned the value of teaching heterogeneous groups of students with the tools of Complex Instruction. I am also the mathematics coordinator for my district, and I am working with teachers to bring these practices into as many classrooms as possible, using the TRU framework as a guide.

Rebecca. In my first year of teaching, I struggled a lot with classroom management with a group of students who challenged me. The second year, I felt more confident. I had gone through the curriculum before and I had only one period of Algebra 1, which allowed more flexibility of pacing. I remember really being able to emphasize conceptual understanding using multiple representations. Students were able to take their time making their tables, graphs, rules, and tile patterns to show how the growth and starting amount were connected across each representation. While I know there were many factors at play, I felt like I was seeing my own success in the students' progress, especially as they continued to be successful in mathematics classes the following years. This year, my eighth as a teacher, I saw another big jump of progress in my teaching. Last spring through this year I've been learning more strategies and ideas through classes, workshops, conferences, and collaboration. Most notably, I've learned about and begun to implement Complex Instruction. The structure of Complex Instruction has made a profound difference in my teaching and my students' learning. It has fixed problems that I didn't even realize were issues until seeing the progress my students have made. I first bought in to using Complex Instruction because it supported my students in group work. However, in learning about and being able to address status issues, I've been able to see formerly low-status students really shine. This is particularly notable in some students who I had in previous years who had seemed to struggle despite my efforts to help them; now with implementing Complex Instruction, those same students are respected and have their classmates truly valuing their ideas and contributions both during group work and in whole-class discussions.

Intentions

Choosing, designing, and implementing curriculum effectively is a large and complex task. Of course, the accuracy of the mathematical content is essential. Beyond that, there are so many different things to consider that, even for a veteran teacher, it is easy to overlook crucial components. Deep examination of the TRU dimensions will provide a reflective teacher with the opportunity to make significant improvements to their practice. The original TRU documents, however, are intended to be a lens into examining a teacher's enacted practice (as they consist of observation and conversation guides). We wanted to develop a TRU-based tool that would be useful for analyzing and selecting curricula, especially for teachers still in their first years of practice.

While TRU provides a comprehensive way of thinking about mathematics instruction, there are certain areas that would benefit from more explicit attention, such as supports for language learners and authentic connections to students' lives and experiences. We therefore added statements about these components within the appropriate dimensions. We relied on the Culturally Responsive Mathematics Teaching (CRMT) framework (Aguirre et al., 2012) for ideas in these areas.

In designing this tool, we include a Goal (on the left) that embodies a curricular need and Checkpoints (on the right) that are statements about how the curriculum can address the need. A rapid first-pass could darken (or not) the checkboxes—as we have done in the example in the Appendix. A more careful pass might include partially filling a box and making notes on what is missing from the curriculum. The checklists should not be considered comprehensive, but rather a jumping-off point for supporting teachers as they consider each idea. We hope that using this tool will help teachers make more informed choices about the curriculum that they implement in their classrooms, facilitate professional learning, and improve the quality of mathematics instruction that students receive.

The Checklists and Their Use

From using these checklist tools, we hope that teachers will be able to get a quick sense of the overall quality of the lesson they are evaluating, and how well it aligns with their intentions. This tool can help teachers sort lessons into one of three categories: can be implemented as the curriculum suggests, needs minor adjustments before implementation, or needs a significant amount of work before implementation. If comparing curricula to make purchasing decisions, we recommend using the checklist on all lessons of at least one full unit per curriculum to get a sense of each in order to effectively compare. If using the checklist to make decisions about materials, examine enough lessons to provide a comparison or synthesis of information across lessons. If using this to compare and choose the best lesson for your needs, then use the checklist to see which lesson is the most ready-to-implement. If using this on a selected curriculum with no alternate options, use this checklist to see what areas need modifications to be improved.

The reader may notice that there is some repetition across the goals in the different dimensions. The five TRU dimensions are, of course, highly interrelated, so it can be difficult to fully separate them. As teachers may choose to focus their curriculum evaluation on only one or two of the dimensions at a time, we tried to make each checklist independently useful, leading to some repetition. As teachers practice evaluating curriculum using these checklists, they will get better at seeing all of the components of effective lessons that span the five categories. We hope more teachers will become aware of and prioritize having all students (1) discuss and share their ideas as well as (2) use students' own cultural backgrounds to enhance understanding. These are the two areas that we found repeatedly lacking in lessons across different curricula, though these areas have a substantial impact on student learning. With more teachers learning to include student talk and cultural relevance in lessons, we hope to see more options to include these pieces in the official curriculum. After presenting the checklists on the next few pages, we return to this idea and offer some questions and strategies that address implementation, once a curriculum is reviewed.

Mathematics

Goal	Formal Curriculum Checkpoints
It is clear how the mathematical ideas for this course develop in this lesson.	❑ Alignments with standards are stated ❑ Connections to standards are explained to teacher ❑ Connections to standards are made explicit to students
Learning intentions and corresponding success criteria are clearly established (for **math content and practices**; see Hattie, 2012).	❑ Clear indication of the lesson's key math content (i.e. concepts and procedures) ❑ Connections to math **content standards** are explicitly stated ❑ Connections to math **practice standards** are explicitly stated ❑ Success criteria are presented in a student-friendly way (i.e. similarly to how they would be found on a rubric)
Lesson connects with what students already know.	❑ Requisite prior knowledge is clearly indicated ❑ Lesson explicitly builds on students' prior knowledge. ❍ with new modification ❍ as an extension ❍ by connecting multiple ideas
Lesson notes indicate how student difficulties with prior concepts or procedures may emerge.	❑ Potential problematic areas are detailed in teacher notes ❑ Suggested modifications are provided
Students interpret, create, problem-solve, or communicate with multiple representations of mathematical ideas.	Students are expected to use the following representations or to make explicit connections between representations: ❑ Physical (manipulatives, realia) ❑ Visual (graphs, patterns, diagrams, video, images) ❑ Symbolic (expressions, equations, notation) ❑ Verbal (using math vocabulary) ❑ Contextual (real-world situations)
Students engage in mathematical proof and validation.	❑ Students work on aspects of proof and argumentation, such as: ❍ Examples vs. counterexamples ❍ If-then statements ❍ Always/sometimes/never statements ❍ Conjecturing and testing ❑ Students are asked to ascertain whether an idea is true ❑ Students are asked to convince others that an idea is true ❑ Students work on formal mathematical proof (logico- deductive, inductive, proof by contradiction, etc.)

Cognitive Demand

Goal	Curriculum Checkpoints
Students have opportunities to make sense of mathematical content.	❏ Problem-based/inquiry-based lesson ❏ Students discuss emerging ideas in small groups or as a class ❏ Students engage in written reflection about emerging ideas ❏ Productive struggle is planned, with appropriate supports
Students have opportunities to engage in mathematical practices.	❏ Students work on non-routine problems that have multiple solution pathways ❏ Students must reason about quantities and relationships ❏ Students build, communicate, and critique arguments about mathematical ideas ❏ Students engage in mathematical modeling ❏ Attention is given to specific mathematical tools that are appropriate for the given task ❏ Explicit attention is given to precision ❏ Students reason about the structure of problems, connections between mathematical representations, or compare/contrast with previously learned ideas ❏ Students use patterns to make sense of a problem, or to generalize
According to pacing information, students are given an appropriate amount of time for each task.	❏ An appropriate amount of time is allocated for each task ❏ New times are specified if including any modifications suggested by the lesson notes Note: Too little time may cause students to feel pressured or rushed to come to a conclusion. Too much time may lead to unproductive struggle or loss of focus.
There is an appropriate level of cognitive demand in lesson activities.	❏ Students spend time working on tasks of all types of cognitive demand (variety) ❏ Students spend sufficient time working on higher-level demand tasks (challenge) Note: For more on types of cognitive demand (e.g., *memorization*, *procedures without connection*, *procedures with connections*, *doing math*) see Smith & Stein (1998).
Students are supported in sense-making, yet the intended level of cognitive demand has been maintained.	Lesson notes provide: ❏ anticipated misconceptions and ways to address them ❏ differentiated learning supports, in general ❏ specific supports for language learners and students with special needs ❏ participation structures for small-group work Lesson utilizes: ❏ non-routine problems ❏ low-floor, high-ceiling problems (everyone can access the problem and there are built-in opportunities for extension)

(Cognitive Demand checklist continued on the next page)

Cognitive Demand (continued)

Goal	Curriculum Checkpoints
Students are given a chance to explain things (not just provide answers).	❑ Students are expected to explain their ideas or processes: ❍ in writing ❍ in pairs or small groups ❍ to the whole class ❑ Students are expected to justify or defend their reasoning ❑ Students are expected to critique the reasoning of others ❑ Students are expected to revise their explanations based on feedback
Students are held accountable for high-level products and processes.	❑ Students engage in significant written explanation/reflection about their work ❑ Students present their work to the class or in small groups ❑ Groupwork tasks include both an individual and a group product that can be assessed

Note: Equity and Access Checklist begins on the next page.

Equity and Access

Goal	Curriculum Checkpoints
All students get to participate in mathematics learning in meaningful ways. Students cannot hide or be ignored.	❑ Lesson activities provide students an *opportunity* to participate meaningfully ❑ Lesson activities make it hard for a student to opt out of meaningful learning ❑ Lesson activities *require* all students to participate meaningfully (for example, groupworthy tasks where the group can only succeed if all members of the group are engaged)
Students are kept engaged with the material.	Lesson includes: ❑ Content that connects to real-life problems or experiences (i.e. mathematical modeling prompts) ❑ Movement ❑ Games / puzzles / challenges ❑ Problem-solving ❑ Communication activities ❑ Reflection activities ❑ Meaningful student choice
The lesson supports students' academic language development, and draws on their linguistic funds of knowledge (Aguirre & Zavala, 2013)	❑ Text includes explicit supports for language learners, such as sentence frames or Math Language Routines (Zwiers et al., 2017) ❑ Lesson explicitly asks students to contribute language from their personal experiences, such as words/forms of communication from: ❍ other classes (e.g., science, English) ❍ outside of school experience ❍ home language(s)
Learning intentions and corresponding success criteria are clearly established (for language and social skills development).	❑ Key general academic and domain-specific vocabulary words are specified (e.g., Tier 2 and Tier 3; Beck, 2002) ❑ The lesson explicitly describes and supports norms for: ❍ Cooperation ❍ Communication ❍ Active listening ❍ Giving and receiving help ❍ Building on others' ideas ❑ Language and social skills development have clearly stated learning intentions both in teacher notes and student-facing text
Lesson activities are structured to promote equity of voice and participation.	❑ Students have individual think/work time before being asked to speak ❑ Students are expected to build on each other's ideas ❑ The text supports the use of the five practices for orchestrating classroom discussions (Smith & Stein, 2018; Smith et al., 2009). ❑ Lesson notes provide structures to support equity in both small group and whole class discussions
Lesson activities are designed in a way to mitigate/disrupt status issues in the classroom.	Lesson activities provide opportunities for students to: ❑ Demonstrate mathematical competence in multiple ways ❑ Collaborate rather than compete ❑ See the value in examining others' ways of sense-making ❑ Celebrate the work of others

Agency, Identity, and Ownership

Goal	Curriculum Checkpoints
Students have opportunities to generate and share their own ideas. Students' ideas are built upon.	❑ Includes questions with no single "right" answer ❑ Includes directions to "generate", "justify", or "show how you know" ❑ Students are asked to convince or persuade others of their own ideas ❑ Students present work to the class ❑ Students are asked to apply a classmate's strategy to a new problem/situation
Students have opportunities to construct new understandings of their identities as math-doers.	❑ New and different ways of thinking/doing math are highlighted by the text ❑ Students are asked to reflect on connections made and what they contributed to the lesson
Students are recognized as being capable and able to contribute.	❑ Lesson activities allow for student choice ❑ Lesson activities are not over-scaffolded ❑ All students are expected to contribute ❑ Students expected to share more than just computational answers
This lesson helps students connect mathematics with relevant/authentic situations in their lives.	Students are asked to: ❑ Provide examples from their experiences that illustrate mathematical concepts ❑ Research related situations ❑ Pose/solve new problems based on personal experiences ❑ Engage in modeling to solve real-world problems ❑ Use math to understand/critique/change an important equity or social justice issue
Students are expected to assess themselves in this lesson.	❑ Written reflection ❑ Self-assigned score on a rubric ❑ Peer assess/document what was contributed by group members ❑ Focus on growth over time

Formative Assessment

Goal	Curriculum Checkpoints
Formative assessment strategies are present in the lesson.	❑ Specific strategies, like exit tickets or use of individual whiteboards, are provided by the text ❑ Teachers are told what to look for in partner/small group activities, such as card sorts, and suggestions for responding ❑ Technology tools that can be used to check for understanding are included ❑ Teacher text indicates likely misconceptions and provides suggestions for how to respond
Student ideas, including non-standard conceptions, are used to inform subsequent parts of the lesson.	❑ Ideas are used to build future strategies ❑ Ideas are discussed to highlight pros and cons of different strategies ❑ Non-standard conceptions are highlighted for further discussion/investigation ❑ Problematic conceptions are turned into “find the error” problems ❑ Lesson activities branch, depending on outcome of prior activities
Text provides a way to assess students’ work in collaborative pairs/groups as well as their emerging mathematical ideas.	❑ Rubrics, checklists, or reflective activities connected with collaborative work are included

Conclusion

As experienced teachers, we tend to jump into curriculum analysis by heading for the first meaty mathematics. To illustrate the use of the Equity and Access checklist, we took a look at Unit 2, Lesson 1 from the Grade 7 *Illustrative Mathematics* curriculum. In the Appendix we include our version of a completed checklist for the Equity and Access dimension for the lesson. What we noticed included the ways that “hands on” experiences for students were called for in lesson implementation. More generally, once you have identified a reasonable curriculum, there are some questions that are important to address to prepare for an implementation that is equitable and responsive to students.

We close with important overarching questions related to the three TRU dimensions that may be the least familiar to new teachers: equity and access; agency, identity, and ownership; and formative assessment. These dimensions focus on the nuances of implementation required for establishing, maintaining, and assessing the effectiveness of instruction for every student. These questions are a purposeful enrichment of the TRU framework with ideas from Aguirre and Zavala’s (2013) lesson analysis tool for making responsive teaching explicit.

Equity and Access: Implementation Questions and Strategies

- How do you know you established norms in your classroom that will support students’ ability to work productively at the desired level of cognitive demand in the lesson?
- How will you make sure when teaching with worthwhile, challenging, and engaging problems/materials that you do not unintentionally lower the intended cognitive demand (e.g., by over-scaffolding lesson content)?
- Have you considered specific scaffolds of student participation? How do they provide all students, individually and collectively, access at the intended level of cognitive demand?

Strategies to consider in answering these questions:

- Learning about Complex Instruction is useful (e.g., complexinstruction.stanford.edu).
- From Complex Instruction come tools for teaching that puts students in pairs or small groups to support individual understanding, including defined roles and responsibilities, providing language supports for students, and using both individual and group accountability measures.
- Attuning yourself and students to status issues in the classroom and learning to intentionally disrupt them is of utmost importance. Too often, group work fails because of insufficient use of participation structures, lack of plans for student accountability, and status differentials among students.

Agency, Identity, and Ownership: Implementation Questions and Strategies
Classroom discussions are an integral part of robust teaching, but there are some important things to consider to ensure successful implementation.

- How will you decide who is selected to talk during the whole class discussion?
- How can you ensure that a variety of student strategies will be surfaced, and not just those who are more vocal?
- What steps can you take to interrupt status problems as they emerge, and to head them off by making instructional choices?

Strategies/resources to consider:

- During work time, find students with several different strategies to have them share out
- Pay attention to good strategies from lower status students and have them share their ideas (be sure to reinforce the good thinking to assign competence)
- After partner/small group work, use equity cards or sticks to call on students randomly (not taking volunteers)
- After group work, have one person share out from each group (students should know how share out will work: student can be chosen randomly or use process to select as "reporter")

In addition to getting students to participate equitably, they also have to be taught how to listen to and value each other's contributions.

- What strategies will you use to ensure that students actively listen to each other and seek to build on each other's ideas?

Strategies/resources to consider:

- Students work together on strengthening each other's ideas
- Students provide each other with feedback to make revisions
- Student partner talk routines that include responding each other (beyond announcing ideas to each other, *responding* to what the other person has said)
- Teacher facilitates students repeating what they heard before building on a previous person's idea

When student discourse is a large part of a lesson, you have to be prepared for students to take things in unanticipated directions. Are there some points in the lesson where students' emerging ideas have a significant potential to take the class in an unanticipated mathematical direction? What are your decision-making routines for when you follow their lead in the moment, and when you save their ideas for another time?

Formative Assessment: Implementation Questions and Strategies Related to the last point above, mindful planning of formative assessment can give you the instructional think-time you may need to process what is happening in the moment and decide on diverging from your intended direction?

- How will you respond? What will be a few "go to" formative activities?
- Does instruction respond to students' ideas and help them think more deeply?
- How will you give feedback to students on their work, in a way that is timely, meaningful, and yet manageable?
- How will students be expected to act upon the feedback that they are given?

Strategies to consider:

- Follow-up questions for students to provide explanations
- Partners critique each other's reasoning
- "Gots" and "needs" poll where each student writes one thing they now know/think based on what they are learning and one thing they are confused by or need to understand (better).

Author-Recommended Resources for More Information

For more information on some of the ideas referenced in the checklist, we encourage you to seek out these resources on the following topics:

(1) *Math Language Routines* from the SCALE project at Stanford (Zwiers et al., 2017)
(2) Complex Instruction:
  - *Strength in Numbers* book (Horn, 2012)
  - *Designing Groupwork* book (Cohen & Lotan, 2014)
  - SFUSD Website

(3) An overview that introduces many valuable ideas, definitely a good read for newer teachers: *Visible Learning for Mathematics, Grades K-12*.
(4) Smith and Stein's guidebook: *Book Preview: 5 Practices for Orchestrating Productive Mathematical Discussions*, see References, below, for related articles.

Acknowledgements

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Appendix

1.2: Mystery Mixtures (15 minutes)

CCSS Standards

Building On

- 6.RP.A

Building Towards

- 7.RP.A

Routines and Materials

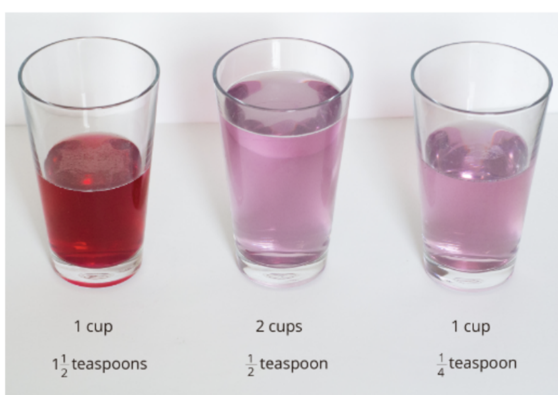
Instructional Routines

- MLR2: Collect and Display
- MLR7: Compare and Connect
- MLR8: Discussion Supports

Required Materials

- Colored pencils
- Drink mix
- Measuring cup
- Measuring spoons
- Mixing containers
- Small disposable cups
- Water

Activity	The purpose of this activity is for students to articulate that the taste of the mixture depends both on the amount of water and the amount of drink mix used to make the mixture. Ideally, students come into the class knowing how to draw and use diagrams or tables of equivalent ratios to analyze contexts like the one in the task. If the diagnostic assessment suggests that some students can and some students can't, make strategic pairings of students for this task.
Launch	Show students images of the drinks.



If possible, give each student three cups containing the drink mixtures.

Tell students to work through the first question and pause for a discussion. Ask questions like,

- "What does it mean to say that it has more drink mix in it?"
- "Imagine you take different amounts of the two that taste the same. There will be more drink mix in the larger amount, but it will not taste different. Why is that?"

The goal is to see that in the same quantity of each mixture (say a teaspoon), the more flavored drink mixture has more drink mix for the same amount of water. (Alternatively, we can say the more flavored drink mixture has less water for the same amount of drink mix.) Use MLR 8 (Discussion Supports) by making gestures or acting out facial expressions for "strength" of the mixture.

After the students have made some progress understanding this idea, the class should continue to the second question. If students finish quickly, press them to find the amount of drink mix per cup of water in each recipe, thus emphasizing the unit rate.

Access for English Learners

Conversing, Writing: MLR2 Collect and Display. Before students begin writing a response to the first question, invite them to discuss their thinking with a partner. Listen for vocabulary and phrases students use to describe how the amount of water and the amount of drink mix affects the taste of the mixture. Collect and display words and phrases such as "more drink mix," "more water," "tastes stronger/weaker," etc., and then encourage students to use this language in their written responses, and during discussion.

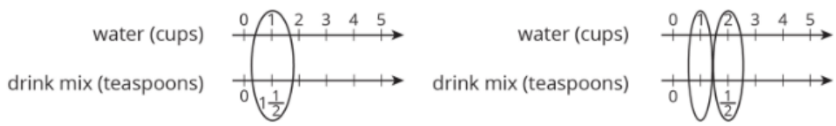
Student Facing	Your teacher will show you three mixtures. Two taste the same, and one is different. 1. Which mixture tastes different? Describe how it is different. 2. Here are the recipes that were used to make the three mixtures: • 1 cup of water with $1\frac{1}{2}$ teaspoons of powdered drink mix • 2 cups of water with $\frac{1}{2}$ teaspoon of powdered drink mix • 1 cup of water with $\frac{1}{4}$ teaspoon of powdered drink mix Which of these recipes is for the stronger tasting mixture? Explain how you know.
Student Response	Teachers with a valid work email address can click here to register or sign in for free access to Student Response.
Student Facing	**Are you ready for more?** Salt and sugar give two distinctly different tastes, one salty and the other sweet. In a mixture of salt and sugar, it is possible for the mixture to be salty, sweet or both. Will any of these mixtures taste exactly the same? • Mixture A: 2 cups water, 4 teaspoons salt, 0.25 cup sugar • Mixture B: 1.5 cups water, 3 teaspoons salt, 0.2 cup sugar • Mixture C: 1 cup water, 2 teaspoons salt, 0.125 cup sugar
Student Response	Teachers with a valid work email address can click here to register or sign in for free access to Extension Student Response.

Activity Synthesis

The key takeaway from this activity is that the flavor depends on both how much drink mix *and* how much water there is in the mixture. For a given amount of water, the more drink mix you add, the stronger the mixture tastes. Likewise, for a given amount of drink mix, the more water you add, the weaker the mixture tastes. To compare the amount of flavor of two mixtures, when both the amounts of drink mix and the amounts of water are different in the two mixtures, we can write ratios equivalent to each situation so that we are comparing the amount of drink mix for the same amount of water or the amount of water for the same amount of drink mix. Computing a *unit rate* for each situation is a particular instance of this strategy. Make these ideas explicit if the students do not express them.

If students do not create them, draw discrete diagrams like this:

Or double number line diagrams like this:



For each mixture, identify correspondences between the discrete and number line diagrams, and between the diagrams and tables:

water (cups)	drink mix (teaspoons)
1	$1\frac{1}{2}$
2	3

water (cups)	drink mix (teaspoons)
2	$\frac{1}{2}$
1	$\frac{1}{4}$

Ask questions like, "On the double number line diagram we see the 1 to $1\frac{1}{2}$ relationship at the first tick mark. Where do we see that relationship in the double tape diagram? In the table?"

Use MLR 7 (Compare and Connect) for students to compare methods of how they knew which recipe was strongest. Who used multiplication? Who used division? Who used a unit rate of water per drink mix teaspoon? Who used a unit rate of drink mix per water cup?

Access for Students with Disabilities

Representation: Internalize Comprehension. Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, use the same color to illustrate correspondences between the number line diagrams and ratio tables for each mixture.
Supports accessibility for: Visual-spatial processing

Equity and Access

Goal	Curriculum Checkpoints
All students get to participate in mathematics learning in meaningful ways. Students cannot hide or be ignored.	■ Lesson activities provide students an *opportunity* to participate meaningfully ■ Lesson activities make it hard for a student to opt out of meaningful learning ❑ Lesson activities *require* all students to participate meaningfully
Students are kept engaged with the material.	Lesson includes: ■ Content that connects to real-life problems or experiences (i.e. mathematical modeling prompts) ❑ Movement ❑ Games / puzzles / challenges ■ Problem-solving ❑ Communication activities ■ Reflection activities ❑ Meaningful student choice
The lesson supports students' academic language development, and draws on their linguistic funds of knowledge (Aguirre & Zavala, 2013)	■ Text includes explicit supports for language learners, such as sentence frames or Math Language Routines (Zwiers et al., 2017) ❑ Lesson explicitly asks students to contribute language from their personal experiences, such as words/forms of communication from: ❍ other classes (e.g., science, English) ❍ outside of school experience ❍ home language(s)
Learning intentions and corresponding success criteria are clearly established for language and social skills development.	■ Key general academic and domain-specific vocabulary words are specified (Beck, 2002) ❑ The lesson explicitly describes and supports norms for: ❍ Cooperation ❍ Communication ❍ Active listening ❍ Giving and receiving help ❍ Building on others' ideas ■ Language and social skills development have clearly stated learning intentions both in teacher notes and student-facing text
Lesson activities are structured to promote equity of voice and participation.	■ Students have individual think/work time before being asked to speak ❑ Students are expected to build on each other's ideas ❑ The text supports the use of the five practices for orchestrating classroom discussions (Smith & Stein, 2018; Smith et al., 2009). ❑ Lesson notes provide structures to support equity in both small group and whole class discussions
Lesson activities are designed in a way to mitigate/disrupt status issues in the classroom.	Lesson activities provide opportunities for students to: ■ Demonstrate mathematical competence in multiple ways ❑ Collaborate rather than compete ■ See the value in examining others' ways of sense-making ❑ Celebrate the work of others

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Research to Practice Sampler

Sustained Support for Teaching Assistants

Sean P. Yee and Kimberly Cervello Rogers

Abstract. Research to practice samplers provide a short review of the research literature on a topic and then offer some examples of professional learning activities that leverage the research. In this case, the topic is the preparation of graduate students for teaching college mathematics.

Continued teaching support for graduate students is critical as they progress towards and throughout their future careers, yet challenging due to the variations and transitions they encounter. Graduate students engage in various means of teaching, including assignments as teaching assistants (TAs, including recitation leaders, graders, emporium instructors, or tutors) and often progress to graduate student instructors (GSIs, where graduate students are instructor of record). Supporting graduate students' learning to teach while they are also responsible for teaching undergraduate mathematics students requires sustained professional development (Yee & Rogers, 2017, Rogers & Yee, 2018).

Research in sustainable professional development (PD) has found that certain teaching practices, known as generative teaching practices (Franke, et al., 2001), endure and continue to grow with novice instructors as they develop their teaching. Franke et al. found that the most pervasive generative teaching practices focused on student-centered instruction, documented through the ability of teachers to explain how a student perceives, thinks, and solves mathematical problems as well as how to create and modify tasks to enhance student understanding. Student-centered instruction differs from traditional TA training programs because the focus is on how students perceive, think, and solve rather than TAs' ability to present material (Belnap & Allred, 2009). Thus, sustained support should incorporate a focus on student-centered instruction to help TAs and GSIs continue to grow as effective instructors.

Fostering student-centered instruction among TAs and GSIs requires both a specific training focus and also buy-in from the graduate student culture. Specifically, a community of practice (Wenger, 1998) around student-centered instruction is needed to nurture graduate student teaching. There is limited literature at the university level about the impact of ongoing PDs with novice instructors (Speer et al., 2010), but there are many ways in which PDs can grow a community of practice. We describe two research-supported methods that can encourage this type of graduate student teacher development.

First, at some institutions, graduate students begin as TAs and then become GSIs, and at others graduate students are assigned as GSIs in their first semester in graduate school. These transitions to GSI can be jolting for graduate students (Rogers & Yee, 2017). To help them

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transition, iterative Japanese lesson study can provide an opportunity for graduate students to collaboratively work together to teach a lesson, ideally during a semester course or seminar prior to the first semester they are GSIs. Lesson study is where multiple teachers work together to design and teach a lesson with a new teaching method. They then use formative assessment to analyze student understanding, discuss revisions, and reteach the lesson with a different instructor to see how their modifications affected student learning. Lesson study has been shown to be useful with graduate students transitioning to GSIs because it provides them a safe and supportive environment to try teaching methods around student-centered instruction (Yee & Rogers, 2016). Moreover, discussion among graduate students about how to teach a course naturally opens a dialogue around teaching, a necessity for establishing and sustaining a community of practice. After participating in an iterative lesson study process, GSIs can reuse the lesson study model to continue to grow with other new teaching methods that focus on student learning throughout their career.

Second, after graduate students become GSIs, continued support through observation cycles has demonstrated positive development towards student-centered instruction. Observation cycles (observation followed by post-observation feedback repeated multiple times) are used in primary schools (e.g., part of a coaching cycle, Gibbons & Cobb, 2017) and secondary schools (e.g., part of mentoring and induction, Portner, 2005). The number of observation cycles members of a mathematics department can provide TAs or GSIs is often limited by resources, but it is important that any observations include post- observation formative feedback—feedback communicated to the instructor intended to modify their thinking or teaching practices (Shute, 2008, p. 154). Similar to the value of formative assessments of student learning, formative feedback is important because it provides formative assessment of teaching. We have found similar results through a peer-mentoring model where experienced GSIs mentor and observe novices (Rogers & Yee, 2018; Yee & Rogers, 2017; Yee et al., 2019). Results suggests observation cycles should include post-observation discussions focused on specific areas of improvement, not evaluations. Specifically, formative feedback for novices should focus on student-centered (generative) teaching practices as discussed in the Mathematical Association of America (MAA) *Instructional Practices Guide* (2018).

Activities for Professional Learning About Teaching

Note that these are activities for which *the learners are people who teach mathematics.* The tasks might be used in a workshop or seminar for novice college mathematics instructors (e.g., graduate students learning to teach undergraduate mathematics). Each activity is based on resources that are publicly available.

Activity 1: Lesson Study Project for Student-Centered Instruction. Materials available at `https://seanpyee.wixsite.com/professional/resources` (Yee & Rogers, 2016)

Activity 1 Goals: Participants will:

- Collaboratively design and teach, revise, and reteach a lesson.
- Engage in the lesson planning process with a focus on measurable goals, formative assessment, and student-centered instruction.
- Reflect upon early teaching experiences and have open communication about how to revise teaching a lesson.

The purpose of Activity 1 is to provide a safe, supportive, and collaborative environment to prepare and teach lessons using the pedagogical content knowledge. Graduate students working as TAs or instructors of record will work in groups (no bigger than three) to (1) design a lesson plan with measurable goals with a focus on collecting student feedback from mathematical tasks, (2) teach the lesson, (3) revise the lesson according to the student feedback, (4) teach the lesson to a different class, (4) write a reflection on how the project informed their understanding of teaching. Only one graduate student needs to teach each lesson, but all group members must attend the both lessons and help in all steps of the process. The same graduate student need not teach the second lesson.

In departments where large lecture courses are taught by instructors who coordinate with TAs as recitation leaders, the preferred method for the lesson study is to have the recitation TA ask their lecturer if they can teach their class for one class period. In departments without recitation sections, the professional development provider should contact faculty members to arrange the time and place for the lesson study group to teach a faculty member's class. In either scenario, lining up a second class time soon after the first is helpful for providing a way for the group to reteach their revised lesson and learn from the iterative process. This activity aligns with the ideas provided on the previous page because it can help graduate students transition to GSIs and build a community of practice around open discussion of lesson planning, teaching and collaboratively solving curriculum problems.

Activity 2: Observation and Post-Observation Feedback. Available in the MAA CoMInDS Resources collection, the Protocol and Feedback Form is an already formatted document. Note: To access it, sign up for a free MAA Connect account (membership in MAA is *not* required).

Activity 2 Goals: Participants will

- Use an observation protocol designed to focus on providing feedback with three sections, teacher, student, and lesson. All three revolve around student-centered instruction.
- Provide post-observation feedback formatively via Red-Yellow-Green comments that can be used for multiple observation cycles to showcase improvement.
- Describe specific active-learning techniques observed and prescribe active-learning techniques to improve student engagement.

The purpose of Activity 2 is to provide a graduate student instructor observation protocol (GSIOP) and a post-observation Red-Yellow-Green (RYG) feedback structure that promotes student-centered, actionable feedback. The GSIOP's cover page focuses on large and important ideas specific to GSIs. The student-, teacher-, and lesson-sections of the observation form have been developed to allow the observer to identify student engagement by attending to the nature of peer-to-peer, instructor, and course materials' interactions around mathematics content. The post-observation feedback is referred to as the Red-Yellow-Green (RYG) feedback and is designed to provide formative feedback for the GSI with manageable size units. The tool was designed to aid the observer in identifying what feedback is helpful for the GSI for continued growth in teaching. The RYG feedback has been designed to be formative and actionable so that the GSI can continue to develop teaching practices that can be referenced in future observations. Ideally, the GSIOP-RYG feedback observation cycle should be occur three or more times in a single semester to support ongoing teacher growth and development.

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Journal of the California Mathematics Project
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Book Review

Common Core Dilemma

Review of: Mercedes Schneider (2015). *Common Core Dilemma – Who Owns Our Schools?* Teachers College Press.

Mercedes Schneider's Common Core Dilemma – Who Owns Our Schools? (2015) is a fascinating and alarming book. The book disabused me of my own naïveté, not only with regard to the common core but also more generally about how things are done nationally. The book includes a great deal of history.

Before the Soviet Union launched their spaceship Sputnik in October 1957, mathematics in American schools was primarily "the third R" in the commonly described "three Rs" of "reading, [w]riting, and [a]rithmetic." For advanced students, mathematics included some algebra, geometry, and trigonometry. More generally, US K-12 education was fundamentally the same as before World War 2. America's pathetic initial attempts in the late 1950s to match the Soviets led eventually to a broad effort to strengthen the US military, the science and technology to support it, and the education system, especially in science and mathematics, which was seen as the foundation for it all. The stage was set for addressing education, once strictly a state and local matter, as a national issue.

The Conant Report, issued in 1959, urged the consolidation of small high schools into larger ones that could offer three years of science, mathematics, and foreign languages. Mathematicians such as Patrick Suppes, Max Beberman, Robert Davis, W. W. Sawyer, and Ed Moise got directly involved in K-12 education. The National Science Foundation supported efforts through the School Math Study Group (SMSG) and summer institutes for mathematics teachers.

These efforts had uneven results. Shifting the emphasis from paper and pencil arithmetic, the SMSG text books sparked a major change, even as they were widely criticized for their heavy use of set notation and emphasis on non-decimal numeration. The "New Math" upset many parents who did not understand what their children were being taught. The chorus of Tom Lehrer's song, *New Math*, captured their frustration:

> Hooray for new math… new math
> It won't do any good for you to review math.
> It's so simple, so very simple,
> That only a child can do it!

The call to go "Back to Basics" arose in reaction to New Math. It emphasized memorization, rote, and skill and drill. It appealed to conservatives who feared a liberal agenda that emphasized

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questioning and imagination. The National Council of Teachers of Mathematics (NCTM) and others opposed Back to Basics by emphasizing higher order thinking skills (HOTS) and relying on calculators to obviate the need for facility with arithmetic. Patricia Ann Wagner, in her doctoral dissertation of 2014, shows that these "math wars" about mathematics education continued in much the same way through the ensuing decades.

Schneider documents the growing federal involvement in education clearly:

> In 1965, the Elementary and Secondary Education Act (ESEA) allocated federal dollars for education, for school libraries, for education research, and more. Later, the ESEA was amended to fund bilingual education (1968) and it was reauthorized in 1972, 1978, 1983, 1989, and 1994. The 1994 reauthorization introduced the idea of "a core of challenging state standards" for all students and included the expectation of curricular "alignment" –and associated, state-determined assessments. (p. 9)

Schneider documents further developments, in which state governors began to collaborate on education, and big business got into the act and seized the initiative. A 1995 keynote talk to the National Governor's Association by Louis Gerstner, head of IBM, cited the then ten-year-old report A Nation at Risk (1983) and demanded "a fundamental, bone-jarring, full-fledged 100% revolution that discards the old and replaces it with a totally new performance-driven system." Gerstner added,

> Without standards and accountability, we have nothing– If we don't face up to the fact that we are the only major country in the world without an articulated set of education standards–and without a means of measuring how successfully we are reaching them, we're lost before we get started.

He said IBM would get the ball rolling with a National Educational Summit with the aim of new standards being in place by the 1996-97 school year. Implicit in Gerstner's remarks was the idea that schools should be run like businesses, with rewards and punishments. This required alternatives for students in failing schools, which in turn led to a claim about the need for market competition in the shape of charter schools and privatization of education.

The National Education Summit of March 1996, held at an IBM conference center in New Jersey, obtained commitments from 40 governors and 49 corporate executives to create a non-governmental organization (NGO) to "serve as a clearinghouse for standards information and benchmarks and public reporting." The resulting NGO was Achieve, and its agenda reflected its corporate financing. Achieve soon began to advise standards writers and evaluate standards developed by states.

The ground was shifting. The 1995 IASA (the reauthorization of the ESEA) called for:

> Alignment of all educational components–curriculum and instruction, professional development, school leadership, accountability, and school improvement, so that every aspect of the education system works together to ensure that all children can attain challenging standards.

It continued:

> Title 1 will ensure greater accountability through the use of state assessments that measure students' progress toward new state standards. The same standards and assessments developed by a state for all children will apply to children participating in Title 1.

The IASA required states to implement standards and tests. However, it had no enforcement mechanism, so many states did not comply. Those pushing toward what later became the Common Core created and worked through a network of NGOs. These included Achieve, the American Diploma Project, Student Achievement Partners, The Education Trust, and the Fordham Foundation (more recently called the Fordham Institute), all mainly funded, sometimes indirectly, by grants from interested businesses. A list of some of that funding, provided by Schneider, is included near the end of this review.

In 1998, Achieve evaluated the educational standards and assessments of Michigan and North Carolina as a pilot study. It judged standards exclusively on the basis of the US Government's National Assessment of Educational Progress (NAEP) program and Trends in International Mathematics and Science Studies (TIMMS) scores, noting "Tests are the critical link between setting standards and holding education systems accountable for achievement."

Achieve sponsored a National Education Summit in 1999, co-sponsored by the Business Roundtable, the Council of the Great City Schools, the Learning First Alliance, the National Alliance of Business, the National Education Goals Panel, and the National Governors Association. The Summit was chaired by Governors Tommy Thompson (Wisconsin) and James Hunt (N. Carolina), Louis Gerstner (CEO of IBM) and John Pepper (CEO of Proctor and Gamble). The conference endorsed the use of tests to measure achievement and the setting of high performance goals on those tests. Achieve held another summit in 2001, at which it reported glowingly on the American Diploma Project (ADP). This was essentially self-praise, because Achieve was a major player in the ADP. The report stated that the ADP would yield

> a common core of English and mathematics academic knowledge and skills, or "benchmarks," that American high school graduates need for success in college and the workforce (Achieve, Inc., *Our History*, retrieved from the web and cited in Schneider, p 35)

ADP aimed to make high school diplomas reflect knowledge and skills of value in the workplace or in college. They assumed the diplomas were otherwise worthless. ADP wanted, in effect, diplomas to be high school exit exams in mathematics and English. Its July, 2008 report, *Out of Many, One: Toward Rigorous Common Core Standards from the Ground Up* stated,

> A critical mass of states has arrived at a common core of standards in English and Mathematics as a byproduct of their deliberate, voluntary efforts to align their high school standards with the demands of college and careers. (quoted in Schneider, p 42)

This report, in effect, lobbied for what became the Common Core State Standards. It avoided calling for national standards, but it suggested that states were, independently, converging on something very much like that in English and mathematics. The "critical mass of states" was never defined but clearly intended to convey the impression of a done deal, even though that was certainly not the case when the report was written.

No Child Left Behind (NCLB) was the 2001 reauthorization of the ESEA. The US Department of Education executive summary stated:

> The new law reflects a remarkable consensus–first articulated in the President's No Child Left Behind framework–on how to improve the performance of America's elementary and secondary schools while at the same time ensuring that no child is trapped in a failing school.
>
> The NCLB Act, which reauthorizes the ESEA, incorporates the principles and strategies proposed by President Bush. These include increased accountability for States, school districts, and schools; greater choice for parents and students, particularly those attending low-performing school; more flexibility for States and local educational agencies (LEAs). (Schneider, page 15)

NCLB required standardized tests to measure adequate yearly progress (AYP) and consequences ("improvement, corrective actions, and restructuring measures") for districts that do not meet AYP targets. The executive summary continued:

> LEAs [local education agencies] must give students attending schools identified for improvement, corrective action, or restructuring the opportunity to attend a better public school, which may include a public charter school, within the district. The district must provide transportation to the new school and must use at least 5 percent of its Title 1 funds for this purpose, if needed.
>
> For students attending persistently failing schools (those that have failed to meet State standards for at least 3 of the 4 preceding years), LEAs must permit low income students to use Title 1 funds to obtain supplemental educational services from the public- or private-sector provider selected by the students and their parents. Providers must meet State standards and offer services tailored to help participating students meet challenging State academic standards.
>
> To help ensure that LEAs offer meaningful choices, the new law requires school districts to spend up to 20 percent of their Title 1 allocations to provide school choice and supplemental educational services to eligible students. (Schneider, pages 15, 16)

Broadly speaking, NCLB was a colossal flop. Untold hours and dollars went to assess AYP and "help" schools that did not measure up. Educational goals, such as those relating to the arts, social studies and civics, that were not measured on standardized tests were downplayed, and some states gamed the assessment system by setting low bars. NCLB was not reauthorized on schedule in 2007-or even by 2012.

Those who saw education tax dollars as business opportunities were already moving. These included opportunities to run charter schools with public funds but no accountability to taxpayers — all while supposedly making schools more accountable! The result was corporate-run chains of charter schools, enthusiastically backed by Jeb Bush and John Kasich, among others. The stage was set for the Common Core.

In its 2007 annual meeting, the Council of Chief State School Officers (CCSSO) recognized the non-uniformity of state standards and turned to Achieve and other organizations with similar points of view to draft what would, in effect, be national standards without any overt appearance of federal involvement.

In summer, 2008 Gene Wilhoit, Counsel of the Chief State School Officers (CSSO) and David Coleman, founder and CEO of Student Achievement Partners (SAP), a national standards-writing company set up as a non-profit, approached Bill and Melinda Gates about funding the of K-12 "common standards." They got the money.

Then in December, 2008, the National Governor's Association (NGA) and the CCSSO issued a report, *Benchmarking for Success: Ensuring U.S. Students Receive a World-Class Education*, funded by the Gates Foundation. Based on this report, chief schools officers of 51 states and territories signed a three-page Memorandum of Understanding in spring, 2009, titled: *The Council of Chief State School Officers and the National Governors Association Center for Best Practices Common Core Standards Memorandum of Agreement*. This MOU states that the CSSO and the NGA own the Common Core State Standards and have the right to modify them as they choose. A state can opt into the Common Core State Standards by the signature of its Chief Education Officer. State legislatures and voters were generally not asked about this.

At last, large companies furnishing educational content and assessments could market nationally. Companies with an inside track, such as Pearson and McGraw-Hill stood to profit beyond imagination. Business supported the Common Core State Standards! To bring about the Common Core State Standards, however, great care was taken in the beginning to avoid any appearance of federal involvement and especially financial control. Voluntary contributions, notably from the Bill and Melinda Gates Foundation, filled the gap. See the appendix (next page) for a sampling of such contributions, as reported by Schneider.

Mindy Kornhaber, Associate Professor in the Department of Education Policy Studies at Penn State, has noted that only 2% of the hundreds of millions that private foundations, including Gates, poured into the Common Core went directly to school districts and 12% of the over one billion dollars spent by the federal government did (these direct funds supported just 1% of all US districts). Most of the money went to non-profits, universities, and non-school entities to build support for the Common Core, in much the spirit of the grants listed above. Readers can only wonder what could have been done with all that money, had it been directed to schools, teachers, and children.

In America today, public schools are struggling, even as many chains of for-profit charter schools, often using public funds, are leading a race to the bottom. Few education professionals know the story of how this situation came about, though one can only wish they knew more of it. *Common Core Dilemma* tells an important part of that story and supplies convincing evidence to back it up. Mercedes Schneider has written an important book, which every education professional and informed citizen should read.

Contributed by: Bob Stein.

Appendix

Examples of Contributions to Dissemination of the *Common Core State Standards* (CCSS)*

Date	Amount	Organization	Purpose
Sept 2009	$2,039,526	The Education Trust	Develop courses aligned w CCSS
Feb 2011	$3,024,695	ASCD	Support to implement CCSS
Aug 2011	$1,400,000	Stanford U	Help teachers implement math CCSS
Nov 2011	$4,463,541	Scholastic, Inc	Help teachers implement math CCSS
June 2012	$4,042,920	SAP	Help teachers understand and implement CCSS
Nov 2012	$1,815,810	Fund for Public Schools	Support NYC Department of Ed integration of CCSS implementation strategies with new forms of teacher professional development to align with emerging functionalities and capacity of Shared Learning Infrastructure
Jan 2013	$3,615,655	Aspen Institute	Support AI's Urban Superintendent's Network, develop resources to integrate CCSS and educator effectiveness practices, use lessons from the field to inform national policy
June 2013	$800,000	National Assn of State Boards of Ed	Support development of a plan for organization and its efforts to provide training and information to implement CCSS
July 2013	$557,168	Harvard U	Support Education Next's work in: Common Core standards and assessments, digital learning teacher effectiveness, and charter schools
Oct 2013	$1,749,070	James B Hunt, Jr Institute for Educational Leadership and Policy Foundation	Support for states to implement CCSS
Nov 2013	$1,383,041	US Chamber of Commerce Foundation	Engage and educate state and local Chambers to support CCSS

*Excerpted from Schneider (2015).