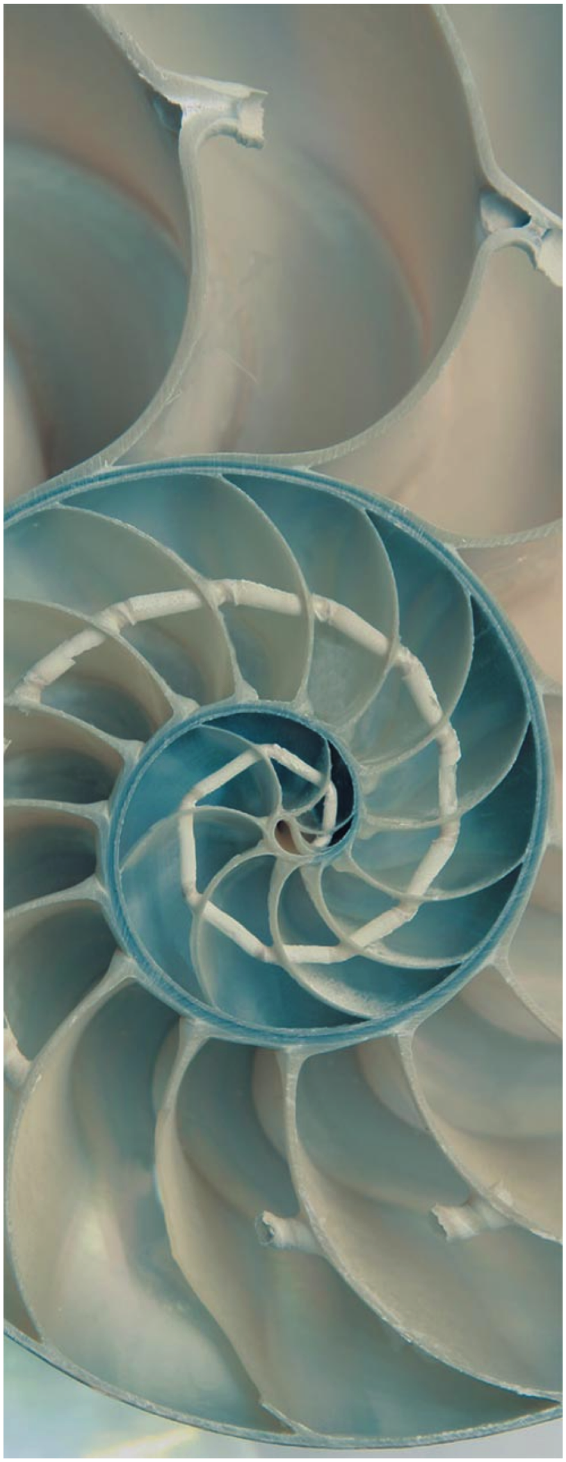

Journal of the California Mathematics Project

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Introduction

The *Journal of the California Mathematics Project* (*JCMP*) is a publication of the California Mathematics Project (CMP) and is sponsored by San Francisco State University. The journal supplements the official news publication of the CMP, *California Online Mathematics Education Times* (COMET), published at ⟨`https://cmpso.org/comet/`⟩.

The journal's mission is communication about mathematics education among those engaged in it, including those active in the CMP or similar initiatives anywhere. Contributions to *JCMP* are made by K-12 teachers, higher education faculty, and a variety of others involved with research and development in mathematics education, such as graduate students and school leaders. The call for submissions is on-going. We do accept simultaneous submissions (copyright is retained by the author). If *JCMP* is the first to accept an article for publication, then the *JCMP* publication must be cited in all other publications of that article, even in revised form.

As can be seen in this issue, the journal publishes a wide array of submissions, including brief research and research-to-practice articles, reports of classroom practice, and book reviews. We also welcome reviews of state adopted materials and insights about programs from authors who have experience with them.

Submission and Review of Material for Publication

Manuscripts are accepted in .rtf, .doc/.docx, and .tex formats, using 12 point Times New Roman or a font with similar size and spacing and with 1.25 inch margins on all sides. Articles are published with LaTeX(a production editor works with authors on formatting). List references at the end of the article in alphabetical order by first author last name with appropriate corresponding citations in the text in American Psychological Association (APA) 7 style. See the articles in this volume for examples of appropriate style and length. For more information and the electronic submission website, please see:
`https://jcmp.calstate.edu`

Submissions will be refereed and may be accepted or returned for revision. Initially, we use a double blind consideration process. To help prepare an article for publication, a referee may reach out to the author to suggest improvement.

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Interview with Judy Kysh
California Mathematics Education: Policy, Projects, and People

Robert Stein

Abstract. Over the last 60 years, the California Mathematics Project (CMP) grew from and was shaped by state policy developments, including revisions to the state's Framework for Mathematics. In this interview, Judy Kysh shares some of her experiences—as a teacher and leader—with mathematics education in California. Judy, who was a co-founder of the Northern California Mathematics Project (which pre-dated the CMP), was interviewed for this article by Robert (Bob) Stein in May, 2023.

Bob: How did you become interested in mathematics education and what were your goals at the time?

Judy: I guess the germ of it was back in elementary school, I was learning math and really enjoying it, right up to third grade, when we learned long division. Our teacher explained how to do long division carefully. He explained it again, slower and a little louder. And then I asked him again, because he hadn't really explained why. Up until then I was one of those fortunate students, I guess, who had been understanding the algorithms we were learning, so this was a shock to me, not to understand it. After repeating the same explanation a third time he moved me from the "fast" group to the "in between" group and gave me extra problems to practice.

That was lurking in the background, but when I was growing up women really had only four choices: We could be teachers, nurses, secretaries or housewives. I didn't want to be only a housewife, though I planned on doing that. I didn't want to be a secretary or a nurse, so I aimed for being a teacher. I found myself in college realizing that I hadn't learned much about writing and didn't like it because I wasn't very good at it - yet. So I ended up taking math. In my junior year the University was funded to create a math major for teachers partly in response to an anticipated shortage and mainly to prepare teachers to use the SMSG [School Mathematics Study Group] materials, the "new math" developed by NSF [National Science Foundation] to compete with the Russians after Sputnik. That's basically how I became a math teacher and got into math education.

From there I taught high school, first in Marin County for 18 years, and I learned a lot because it was the 1960s, and we were using the SMSG materials. You're familiar with those, right?

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Bob: Oh yes, I used SMSG materials when I taught elementary and high school in the early 1960s.

Judy: I really enjoyed the way they developed the reasoning and the understanding. I had a lot of fun teaching and was in a school that was very supportive of professional development and teachers working together. We had a lot of leeway to create our own professional development programs. You've heard of Dr. Stein [Sherman Stein] and Cal Krabill up at UC Davis. Cal who was having students work in groups. We invited Cal to come down and teach us. Then a lot of us started using groups and then actually insisting on new furniture. We had that combination back then of teaching for understanding and building on students working together. A big part of this was a belief that the students had to do the problems themselves and the teachers had to do the problems themselves. We wanted them to be problem solvers.

There was a lot going on in the 1960s, but then, as you are aware, in the early 1970s a back to basics movement began. I think it was SB90 that recreated basic testing. Interestingly, there hadn't been any testing to speak of in the '60s, so we had a lot of freedom to develop new courses, and a lot of freedom with content. We didn't feel free with the academic content that prepared students in college prep courses-we felt beholden to what the colleges and universities required in algebra and geometry, but we created a whole bunch of other courses. We explored geometry in a variety of ways including transformations, alternative geometries, fractals. We created a probability and statistics course as an alternative to second semester of Algebra 2.

Bob: How did your career evolve? Did your goals change over time? What and who were the most important influences in your development?

Judy: My overall goals haven't changed, though I have changed jobs a lot. They still amount to "Math is for everybody" or "More math for more people."

Toward the end of my high school teaching, in the late 1970s, things started going back to basics. At that time I was working with a lot of other teachers in our school developing cross-curricular projects. A social studies teacher and I got invited to the Bay Area Writing Project summer institute. It was their first cross-curricular experiment, inviting teachers from different fields. The English teachers were writing, practicing what they were teaching. I thought at the time we should be doing this in math. We should be having problem solving institutes where math teachers could just get together and work on fun problems-big problems, interconnected problems, and practice problem-solving. We didn't do a lot of that unless we happened to find a problem that we shared with our colleagues. This would actually promote problem solving as an important goal.

At the time I was really ready to move on, because I was seeing everything we had built in our school falling apart. The last straw was when our principal, who knew nothing about teaching math, suggested that I consider teaching more to the test in one of the courses we had developed to promote problem solving and fun or relevant math. It was time to do something else. Before I started teaching I had decided to quit after two years either to go back to school for an MA in math or to teach abroad. Teaching abroad didn't work out, so I applied to graduate school back at UC Berkeley. At the time (1965-66) there were only two women in the graduate math program, and both of us were directed toward MAs, not PhDs. The master's degree turned out to be necessary for university jobs later on. It also reminded me of what it felt like to struggle with understanding the math.

After teaching for 18 years, I worked for a couple of years for the Lawrence Hall of Science, with a program in East Oakland. I worked with teachers and taught at Castlemont High School and Elmhurst Middle School in a program designed to better prepare students graduating from high school to go directly into calculus at UC Berkeley and succeed. At the time, almost all of the students and the majority of teachers at both schools were black. Castlemont was using the same textbook I had used at Drake HS in Marin County, so the preparation was easy, and I was able to hold my students at Castlemont to the same expectations as I had in Marin County. In fact, I told students I had higher expectations for them because they were a more select group. There were 33 students in the class (31 girls/2 boys). These students represented all the 11th grade students considered by the school to be "ready" for Algebra II. At Drake High, where the total number of students in the 11th grade was about the same, we had at least four sections of 35 eleventh graders taking Algebra II. As a much smaller portion of their class the Castlemont group was a more select group.

That was a great learning opportunity for me. When I was at Castlemont or Elmhurst, I was often the only white person. It was an opportunity to learn about how it felt to be conspicuous because of my race. I felt like I had to be very careful not to offend and to figure out how to fit in.

While I was at Lawrence Hall of Science, a job announcement came out from UC Davis for somebody to lead a math project. Three Davis professors, Tom Sallee, Kurt Krieth, and Don Chakerian, were proposing a math project where they would teach math in the mornings, and then in the afternoons the participants would make lesson plans based on the mornings. They were looking for somebody to direct the project. I had been talking about a math project since I had attended the Bay Area Writing Project in 1976. I had been wishing there were a math project that was similar, with a problem solving focus. There were a lot of NSF institutes where people went and took courses, and I never went to those, because they were all about stuff I already knew, courses I had already taken. I thought we needed an institute focused on problem-solving, where the afternoons would be devoted to teachers sharing their best teaching ideas.

I had been talking about this, so when this job announcement came out my colleagues said, "You better put your money where your mouth is and go apply for this job." So I did!

Bob: It was a perfect fit.

Judy: I went to an interview before a large panel, pitched my idea, and they went for it. I got hired as director. We pretentiously called it the Northern California Math Project.

And it was a good fit, because it turned out that Tom was really interested in focusing on problem solving, Don provided a whole new perspective on geometry and its interconnections, and Kurt provided an opportunity for teachers to learn some probability and statistics, an area many math teachers were not familiar with.

Bob: Was that part of the California Mathematics Project?

Judy: No. The California Math Project didn't exist yet. We helped it get started. Our first Davis project was in 1982, and UCLA did something similar, led by Jim Caballero, in 1982 or 83. In those first couple of years Jim and I got together and tried to get people from other UC's and CSU's to consider having math projects. At a meeting at UCLA to drum up interest, some of the people were enthusiastic, but they weren't sure they could handle it. In the meantime I had written a letter to State Senator Gary Hart suggesting that the state of California fund a California Mathematics project similar to the California Writing Project, but focused on problem solving. I think I still have a copy of that letter. I actually went to Sacramento to talk about his writing some legislation for that. Anyway, it came to pass. Funding came through soon after that, and they started handing it out right away. I believe that was in 1983, and the first projects started in 1984. You had one in San Bernardino, right?

Bob: Yes, we were one of the original ones, John Sarli and I. And it was a great success.

Judy: People loved it, and a lot of them really changed their thinking about teaching, to teaching for understanding and problem-solving. And I think that's when Susie [Håkansson] got involved, because there was a transition from Jim to Susie as director at UCLA around that time.

Bob: And you have been at it ever since, I gather.

Judy. That's how I got into it, and I have been working on different aspects of it ever since. In 1985 the UC Office of the President decided they needed a statewide coordinator. I was the first statewide coordinator, and it was half-time.

Bob: Sometime around then the California Math Project was made permanent and would need a full-time coordinator. The UC Berkeley Office of the President invited me and Elizabeth Stage to help with the job announcement. When Elizabeth, and I met with a UC Berkeley VP, I had one goal in mind: To make sure they chose someone with a solid knowledge of math and math teaching. They agreed, and I left satisfied, but then they hired Phil Daro. I was surprised, because he came from the *Bay Area Writing Project.*

Judy: Yes, I actually applied for the full-time job and was a little disappointed when Phil got it. He did a pretty good job of leading the group. I think the project thrived because a lot of good people got involved. That early group also included Julian Weisglass at Santa Barbara, Sharon Ross, Bill Fisher, Carol Langbort, Pam Clute, Carol Frye Bohlin, Nick Branca, Judy Mumme, and more.

Bob: Phil had a fantastic knowledge of the politics and how to keep various groups satisfied, as far as I could tell.

Judy: He did keep a lot of people satisfied but also riled up the group that caused a lot of problems later on. I'm talking about the back to basics group in the 1990s that succeeded in blue-lining the California Math Project in 1998-99.

Bob: I think the worst summer in my life was with the Framework Revision Committee in 1997. I have never seen so many people who should have known better talk past each other. I don't think anyone meant harm, but they didn't listen to each other. One group, supported by Mathematically Correct, insisted on clear delineation of the mathematical content to be taught, while others focused on pedagogy and motivation and disregarded the concerns of the first group. Neither group would listen seriously to the concerns of the other, much less honor them. The meetings were not orderly. The open meetings law only made things worse, because any attempt to hash things on the side would be illegal if it involved three or more people. By summer's end, the framework was nowhere near completion, and everyone was frustrated and angry. It was a perfect storm.

Judy: I was on the '92 Framework committee that set up the problem that the '97 committee was supposed to address. I've been on the '78, '85, and '92 committees. The state does not allow enough funding to support completing the work, so people in the state department of education end up completing the frameworks and resolving any unresolved issues. In '92 the huge unresolved issue was content standards. Prior frameworks had been way too detailed about content, but the '92 framework came out without any content standards at all. Alan Schoenfeld, who was also on the '92 committee, and I both realized that was problematic. We wanted limited content standards. Past frameworks had had way too much detail about what needed to be taught, but you do need *some* content standards, if only because people are used to them and it's sort of their anchor.

Phil Curtis was also on the '92 framework committee, and he, Al Manaster [Math Diagnostic Testing Program group], and I tried to create some content standards that we could sell to the anti-content standards group and that we hoped the people who wanted standards would be happy with. We did this by phone (no Zoom then). I remember conversations, some two hours long, trying to hammer out those standards. They required intense effort and I recall feeling exhausted when we hung up. We came up with something and proposed it, but it never went anywhere. I always felt like if only it had, the whole thing that you described might have been avoided. We knew what was going to happen. You can't just take that [content standards] away from people. It's got to be there.

Bob: The uses of mathematics today are far more diverse than when we grew up, when the main focus was preparation for calculus and beyond. How do you think the K-12 curriculum has been adapted and should be adapted to this new world?

Judy: I think we started to make some progress with NSF projects and state funding for CPM [College Preparatory Mathematics] and IMP [Interactive Mathematics Program]. When Tom Sallee, Elaine Kasimatis, and I proposed the CPM program in 1988, our goal was to work with local teachers to develop more accessible, inclusive college prep math curriculum. While all of the other projects were more ambitious in relation to integrating the content from algebra, geometry, probability and statistics across grades 9-11, we decided that for teachers to be able to use the curriculum within their existing school structures we needed to maintain a focus on preparing students to meet their course by course expectations for Algebra I, Geometry, and Algebra II, but that we could integrate and connect to other areas. For example, by using an area model as a basis for algebraic representation and the distributive property, or using probability problems to build proportional reasoning. We also planned to focus on problem solving, using such strategies as 'guess and check' to grapple with a realistic problem and build tables and patterns to create an algebraic representation. With this compromise on integration and a focus on problem solving we were still able to work with teachers to change their role from "sage on the stage" to "guide on the side."

Throughout the 1990s the program continued to spread teacher to teacher, and we kind of slid under the radar of those who were attacking the "reform programs." In 1992 we formed a non-profit corporation, CPM Educational Program, in order to print and distribute the materials and provide 8 to 10 days of teacher lead professional development for new users. Students in all 50 states are now using the program. We now provide materials for grades 6 through calculus, and our fourth edition will start coming out this year. We use the proceeds from the materials to offer free PD for new users, support our tech and writing groups, provide small research grants and support a group of teacher researchers. Right now, a major focus of our teacher researchers is assessment and grading—the testing we do does not make sense in relation to teaching and learning.

Bob: Finally, Judy, what advice do you have for those contemplating a career, either teaching math or in preparing future math teachers?

Judy: Go for it! We need all the help we can get. There's never a dull moment. If you have a dream, if you have goals, there is always work to be done. I have never regretted entering or staying in this profession, as I am always learning something. Since 2000 I've been at San Francisco State working to prepare future math teachers and returning MA students. Things are always changing, not always in the direction I want, but I think every time we move forward and fall back, at least the next time I hope we will have learned a little more.

Note: For more on the early development of the California Mathematics Project, see the interview with Dr. Susie Hakånsson, the first Executive Director of the California Mathematics Project (JCMP, v10).

About the Author

Bob Stein is Emeritus Professor of mathematics, California State University San Bernardino and editor of the *Journal of the California Mathematics Project* (JCMP). His scholarly work focuses on the preparation and professional growth of future and in-service teachers of mathematics.

Journal of the California Mathematics Project
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Bringing Joy to Uninspired Teachers of Math
Touchstone Strategies, Part 2†

Hal Melnick

Abstract. The second in a three-part series, this resource describes and illustrates three of eight Touchstone Strategies for teacher educators to use in their work with mathematics teachers. The article explores how to inspire teachers to find the joy in mathematics so they can support their students to do the same. Through a variety of tools, techniques, and helpful hints, the eight touchstone strategies in the series illustrate what high quality mathematics instruction looks like and how teachers can reframe their own thinking about mathematics to create deeper learning opportunities for their students. This piece, Part 2, describes the three touchstone strategies: *collaborative math*, *honoring mistakes*, and *journaling*.

The suggested instructional practices I refer to here grow from my life of teaching at Bank Street College of Education, as first articulated in my dissertation research (Melnick, 1992). I aimed to study the nature of change that my students repeatedly told me they experienced while in my classes. I unearthed the themes of change they were experiencing before, during, and up to five years after completing Math for Teachers with me. Four themes were found: (1) grief in a graduate-level course, (2) healing, (3) reconstructing of one's math self-identity, and (4) unpacking their personal locus of change. Key strategies that emerged from that study and that I use in my instruction will be addressed in this work, which is intended as a resource for math teacher educators.

Throughout my career, in student course evaluations and conversations with fellow faculty, I have been described as an effective math teacher educator. I believe there are factors in my teaching that cause that. Most specifically, I believe it is the conscious effort I make to address the emotional component in my teaching that gives rise to these comments. I boldly address the affective realm in my instruction. In this work, I offer teacher educators a set of suggestions to consider when planning to teach groups of teachers who may present as having been taught by being told about math rather than by doing math. I will lay out eight touchstone strategies, behaviors, perspectives, or moves that I have enacted time and again as I taught teachers (see Table 1). Each is designed to help reveal the feelings students have about math and their perceptions of themselves as math thinkers.

Anyone teaching Math for Teachers at Bank Street or a similar course elsewhere may consider some or all of my indicated touchstone strategies. I use the term touchstone since I believe the term best characterizes strategies that include the affective dimension of teaching mathematics. I offer this truncated set of ideas in the spirit with which our annual Bank Street professional appraisals are conveyed: through consistent and self-revealing generative inquiry.

† From an original report by Melnick (2018), this is the second of a three-part reprint with the permission of the author. The full report is available from the Bank Street College of Education website.

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Table 1. Touchstone Strategies*

Strategy	Brief Description
1. Math Autobiography	Use a math autobiography as an in-class assignment.
2. "Do" Math	"Wow" students in the first class meeting by doing lab-type experiential tasks across Nursery School through Grade 6.
3. Collaborative Math	Model how collaborative group work is a special kind of group work.
4. Honoring Mistakes	Model how to honor mistakes and see them as opportunities rather than failings.
5. Journaling	Monitor everyone's learning through a math journal that is linked to class readings.
6. Work a Problem to Death	Work one intentionally perplexing problem to death, unearthing confusions that arise.
7. Non-Dominant Language	Teach one class meeting in a language other than the dominant one.
8. Concept Teaching Games	Have each student plan and share their own concept teaching game.

*The first two strategies are detailed in Part 1 (Melnick, 2022), #3, 4, and 5 are the focus of this report (Part 2).

Touchstone Strategy #3: Collaborative Math

Model how collaborative group work is a special kind of group work.

I always tell the teachers attending my classes that for the first five years I taught children in my Queens, NYC public school, I diligently put kids in groups to work. I expected the children to work cooperatively. There were always difficulties in those groups. Some children took control. Some did not engage and did very little work. And I never could tell who was responsible for the work that got produced. I had no consistent idea about how to avoid those problems and assumed it was what happens during group work. Then, in 1985, I took a teaching math course with Marilyn Burns held in Westport, Connecticut called Math Solutions. I finally understood what was wrong with my planning.

She told us to call these group times "cooperative group problem-solving" and urged three very specific guidelines about how those groups should work. In Math Solutions classes we always worked using Burns' "Three Rules for Small-Group Work" (Burns, 2015, p.113). Suddenly, it afforded me the chance to learn how to listen to my tablemates, how to ask questions I was burning to have answered, and know when and how to ask the teacher for assistance. Ever since then, when I teach children or teachers, I post the Burns rules and discuss each rule with the class (whether adults or children). I always ask, "What does this rule mean or suggest?" and "How can this help you work better as a group?" Burns' "Three Rules for Small-Group Work" are:

(1) You are responsible for your own work and behavior.
(2) You must be willing to help any group member who asks.
(3) You may ask a teacher for help only when everyone in your group has the same question.

I suggest that you, as a teacher of teachers, carefully build your case for why Burns' "Three Rules for Small-Group Work" are essential. They eliminate so many missteps in orchestrating group work, and open opportunities for the emotional aspects of group work to become transparent. But you, the teacher, must be willing to operate by those rules yourself. I found the first few semesters when I introduced the rules to new teachers, I had to go through my own attitude adjustment. I remember how I had to force myself to not answer any question asked of me by new teachers while they were in small groups. My job at those moments was to ask the questioner, “Did you check with your group mates? What did they say?” My teaching of adults improved by leaps and bounds. I was modeling exactly what I hoped they would do with their own classes. Graduates of Math for Teachers tell me that is what happens in their classrooms. And it takes just a little bit of time to impose those rules. Just last week, a teacher told me these group rules are what saved his teaching and how he uses them across curricular areas.

One point I must make: It is essential to emphasize Burns’ Rule #2 “You must be willing to help any group member who asks,” because it communicates there are no stupid questions in math. People suddenly find they no longer censor themselves when they are confused. This rule can undermine the negative self-perceptions that are held by many who believe they cannot do math. Those “old tapes,” if you will, get a chance to be shredded. I urge you to try these three brilliant rules. By doing so, you are actually asking your class groups to solve most of their own problems and communicate their methods to each other. And you, as teacher, learn to more fully trust your students to resolve their own problems. The process is powerful.

I remember reading that Pythagoras, in his youth, played with his buddies using stones and rocks which led him to eventually proving why $a^2 + b^2 = c^2$. Using rocks, pebbles, and straight edges in the sand, he and his buddies supposedly debated whether the squared area of the hypotenuse of any right triangle is really equivalent to the sum of the squared areas of each of the other two legs of a right triangle. By experimenting, he proved his formula beyond a doubt as a pattern that works in every sized right triangle imaginable. Imagine that! For most of our Bank Street candidates, math learning had not happened by doing things, like Pythagoras had; instead, everything was shown to them. They repeatedly write in journals that math became “other-directed” for them, and was never constructed or “owned” by themselves. In the “telling” approach, as I call it, these learners probably never really did mathematics at all. The following quotes from journals flesh out the emotional impact that doing math in collaborative learning groups can have on teachers. Listen to their reflective wisdom when the math work was hard, sometimes causing emotional upheavals and requiring deep math understanding to solve a problem. Listen for the internal satisfaction when things got resolved in the groups. And listen to the new learning that resulted from the use of this imposed protocol.

On feeling supported:

> I definitely felt supported and encouraged along the way in this classroom—by my teachers and my peers. Although group work was sometimes hard, it was a good lesson for me to learn as a future teacher, to acknowledge that group work takes patience and persistence sometimes. It may not always be enjoyable since everyone learns differently. There were times I had working partners who took too much of the lead or were working too fast for me. But there were also peers who were able to explain to me in a different way that was relatable to me. Therefore, it made me realize that all of these same issues will happen with my students, and it’s important to stay on top of that and observe as a teacher. —Erica

On communication in math:

> I appreciated how the approach to solving a problem was emphasized; this was a big part of why I think I achieved my first course goal. Whenever we worked on a problem either as a class or in small groups, I was always asked to explain my thinking. This often made me go beyond my initial strategies to ones I could explain more clearly . . . . This course also helped me generally become a better listener in mathematics. Through small group work and class discussions, I have learned about how to listen to others to

> better understand their strategies. I think this will serve me well when I am working with students in the math classroom. —Anne

On authentic learning:

> It was truly a beautiful experience watching my fellow classmates in many ways let out their inner child when it came to playing the games we paired up for, along with teaming up to solve problems that were presented in front of us. I now possess a refreshed perspective on how mathematics can be instructed, as well as how I can take steps towards becoming the masterful educator I envision in my head. —Eric

On learning how to respect and honor differences:

> I have been able to experience first hand, and reflect on the notion, that everyone has different types of intelligence and areas of strength. This concept was made apparent to me when we worked in groups to solve a perplexing problem, and each team member was able to share how their mind best interpreted the problem. Each one of the group members interpreted the problem differently, and we were all given enough time to conceptualize and explain our method. Another beneficial aspect of this activity was when our professors asked us to reflect on how we contributed to the success of the group. Answering this question allowed us to realize that each person brings something different to the table, but each is equally as valued. —Katherine

On what it really feels like when you personally collaborate to solve a problem:

> Collaboration between educators is something we talk about in all classes in Bank Street, but I have found I learned most about collaborating with a group of people more in this class than in others. I think collaboration clicked in this class because of how fun it became to get to really understand how people were thinking and solving the same problem in different ways. It was a really good metaphor for me about how collaboration can and should happen in other domains as well . —Kelley

On going at your own pace and developing patience with others:

> I learned that everyone needs to go at their own pace. Once I got the answer I wanted to be done, or at least have everyone thinking and feeling the same way as I did, but I realized that some people needed more and it was my job as a team member to give that time. I also learned that problem solving in a group is REALLY rewarding! I felt like by the end of the time, we had really gotten to know each other as learners and teachers. —Abby

If you put teachers in cooperative groups to solve math problems, abide by cooperative learning group rules, and then discuss varied solution strategies, you afford teachers an amazing venue for self-reflection and truly honest learning. With that personal experience, such as I had in 1985 with Marilyn Burns, there is a greater likelihood your teachers will bring the same group methods to their own classrooms of children. Group learning is qualitatively different from cooperative group learning. I urge that all teachers deeply consider the difference.

Figure 1. Seven examples of teachers engaged in collaborative math.

Touchstone Strategy #4: Honoring Mistakes

Model how to honor mistakes and see them as opportunities rather than failings.

Watching “happy” mistake-making has become a joyful and quite surprising enterprise for me as a teacher. As a young teacher/educator, I thought my role was to protect adults from feeling diminished when they gave a wrong answer. I personally remember a kind of humiliation I felt when, as a child, I stood at the blackboard in fourth grade, my back facing my classmates, writing a wrong process to long division on the blackboard! New teachers at Bank Street have often told me the very same story in their journal recollections. They describe how their peers ridiculed them and laughed at them. I did not want to ever let that ever happen in my classes of adults.

But one day in my class in the 1980s, a young teacher made a big mistake describing how she had solved what we affectionately call “The Letricia Problem” (see Touchstone Strategy #6, to appear in Part 3). While sharing, she herself noticed her mistake. With a broad smile on her face, she proclaimed it as a milestone for herself. She made a mistake in front of everyone and she noticed it before anyone groaned! Sheer happiness ensued for her. A light bulb went off for me. She owned her own process and was empowered as a learner. My mindset completely shifted in that moment. Since then, talking about the power of mistake-making in math has taken center stage for me.

And in the book *Making Sense*, which we have used for years in Math for Teachers, the authors point to the social culture of the classroom as a place where mistakes are viewed as healthy instances to learn from. Being transparent with your own mistake affords an opportunity to reconsider and not make the same mistake again. Mathematicians do that all the time while developing new theory. Mistakes sharpen their conjecturing and experimenting. Why not use that as an intentional teaching strategy? A mistake which leads you down a blank alley, gives you important information on the way to getting right answers. When you make a mistake, you can know to never do that again. The mistake is informative. “Mistakes must be seen by the students and the teacher as places that afford opportunities to examine errors in reasoning, and thereby raise everyone’s level of analysis. Mistakes are not to be covered up; they are to be used constructively.” (Hiebert, et al, 2000, p. 9).

Recent research on brain development also has added to our knowledge about making mistakes. Jo Boaler, noted Stanford University professor who today is revered for revolutionizing math education in our country, cites the work of Carol Dweck on fixed and growth mindsets. Boaler and Dweck’s ideas enlarge our understanding of mistake-making even more. “The recent neurological research on the brain and mistakes is hugely important for us math teachers and parents, as it tells us that making a mistake is a very good thing. When we make mistakes our brains spark and grow” (Boaler, 2015, p. 12). Boaler goes on to say: “The various research studies on mistakes and the brain not only show us the value of mistakes for everyone; they also show us that students with a growth mindset have greater error recognition than those with a fixed mindset . . . The ideas we hold about ourselves —in particular, whether we believe in ourselves or not —change the workings of our brain” (Boaler, 2015, p. 13). If you believe you can learn math and do not fear learning math, a mistake is a boon for your brain.

My teaching assistant and I so often watch our class participants twitch their noses and laugh at themselves, saying that prior to their use of manipulative materials, they believed “when you divide one number into another, the answer must always be smaller than the number you are dividing into.” There are three little sixths that can fit on top of the one-half block. So when we ask ourselves this division question, we might ask: “How many sixths can fit on top of the one-half block?” (The answer is clearly three!) *HINT: Use the images of blocks to help think that one through.*

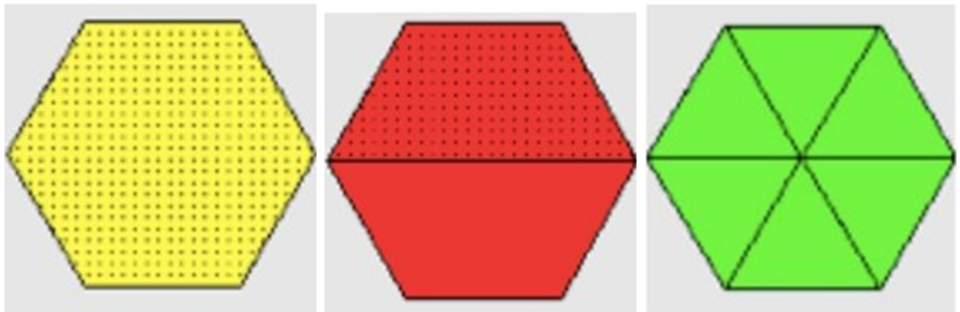

Figure 2. Three hexagons. The yellow can be called one whole. The red can be called one half of the whole. And the green can be called one sixth of the whole.

That is often the moment in our class when people come to see that they learned math solely as a lot of pieces of disconnected, memorized nonsense. Then they often admit they were left with little or no relational mathematical sense. They know to invert and multiply when dividing fractions, but have no idea why that would get a correct answer. By laughing at this absurdity and seeing proof in the manipulatives (e.g., the pattern blocks on their tables), they begin to admit that math is really a study of relationships embedded in "stuff," and they begin a mental change process. There is often delight on their faces.

What I find is that it's quite easy for people to initially change their view of what math really is: a subject that is useful in most everything they do. It is only much later —and only after repeated successes at proving answers to problems with materials —that they start to change their view of themselves as mathematically thinking people. Our Bank Street classroom practices value them as thinking beings. We honor the mistakes made along the process, rethink the language used when mistakes are made, and carefully choose materials that can provide "proofing." They actually see evidence of relationships. They come to own their understanding of those relationships. We then ask people to conjure up generalizations which have grown from their observations of patterns that repeatedly work. Only then do firmer conceptual understandings get planted to replace confusion. Concrete materials often prove the concept to be indisputably true (or not true). Instead of saying "it's wrong" they start saying "that doesn't work." Again and again, students in Math for Teachers find that their early mistakes bring them closer to getting "it right." Being okay with making mistakes allows people to change their perceptions of themselves from receivers of nonsense to "doers" of mathematics. By making mistakes and talking about it, people become thinking/communicating people in the realm of mathematics. I often quote Dorothy Buerk, Emerita Professor of Mathematics and Computer Science at Ithaca College who sees math phobia (or math aversion) as "the process of disintegration when a meaning making organism can no longer make meaning" (Buerk, p 61) . When I quote Buerk's insightful definition, I am invariably met with a huge sigh of relief, especially by the women in my classes.

We also read the *Piaget Primer: Thinking, Learning, Teaching* by Ed Labinowicz (1980). In it, Labinowicz beautifully articulates Piaget's ideas associated with disequilibrium as a necessary component to the learning process. Our job as teachers is not to give people a set of problems which, on first viewing, they can do. In fact, our job is to give teachers problems (situations they understand) where a solution strategy is unknown to them (Burns, 2015). Our intention should be to upset the apple cart; to have adults be confused about where to go with their thinking and stir a need to collaborate with others to try to rethink a line of inquiry and get out of their shared confusion. Piaget called this assimilation and accommodation.

This process often results in learners moving into disequilibrium, which can admittedly feel quite unsettling. We then have to grapple with a thinking process to untangle ourselves by testing hunches and building models or math representations using manipulatives or drawings or equations. We hypothesize new ideas towards a solution. We do this over and over until the pieces of the puzzle all begin to fit together. Together with others we find an agreed-upon logical answer. At that point, we move back into equilibrium. This is the learning process we speak about over and over as Bank Street teacher-educators. Let it happen in your math classes with teachers. Although this is a lifelong cognitive process, I urge you to read the following quotes from our learning teachers and consider how emotions played a potent role. Read how these people expressed a revised appreciation of mathematics and, in some cases, a willingness to regard themselves as thinkers and doers of math themselves.

On learning from mistakes:

> Mainly, I can point to the lack of doing so (making mistakes that is) as a common theme in my education, throughout not only elementary school but through high school. Specifically, I remember a tutor I had for a period of time outside of school. I can't remember precisely how old I was, but it was probably around fifth or sixth grade. I also can't remember her name alas, but I do remember that I liked her, and she was well- intentioned, but I recall that she had a habit of moving on from problems I got wrong and trying a "new" problem of the same ilk, rather than really going over how I got the one at hand wrong. With this dimension in mind, I'm sure I would have benefitted from taking that time to go over my thought process that rendered the incorrect answer. While my tutor experience was technically outside of the classroom, I still feel it bears mentioning since it's so important to view mistakes as learning opportunities as an educator. Within the classroom, I generally recall my math learning being a solitary experience. There was direct instruction, then individual practice, then tests, (with no time to really learn.) —Caitlin

On learning to feel confident to trust students to learn with less of a teacher's intrusion:

> Which brings me to the second part of this reading that stood out to me. Labinowicz writes, "a child's errors are actually natural steps to understanding" (p. 55) . Before I read this piece, I believed that this was true, but Labinowicz's explanation of Piaget's theory helped me to understand why this is the case.It is not just that mistakes are something that happen along the way to finding the right answer, they actually help students find the right answer. Reading this piece gave me the confidence to sit back a bit more when working with my students, to let them make the errors that will promote the natural path of their learning. —Alex

On living with discomfort intellectually:

> While the games and readings helped me with the specific skills, what helped me most was coming to terms with my own disequilibrium. After reading *The Piaget Primer*, I developed a new approach to thinking about my own challenges and discomfort with math. For the rest of the course, I tried to view moments of struggle as learning opportunities. This helped me to push forward and try out new strategies when working on "The Letricia Problem" and "The Perplexing Problem." —Molly

On thinking in different ways:

> I have learned that math should take more than just one approach, and in order to be a great mathematician, you have to have a flexible mind. It was so interesting how all of our brains worked differently to solve one problem, and eventually we all got the same answer, but in different ways! Math is all about communication, and we would have never been able to solve this problem without using visuals, bouncing curiosity and questions off of each other, and taking risks that we may make a mistake along the way. —Abby

And here is another caveat that I urge you to face as a teacher of teachers: If you intend to teach teachers math in a way that is unfamiliar to them, uncomfortable for them, or, for that matter, completely contrary to the way they were taught, you should anticipate a tremendous amount of emotional upheaval and pushback. I have come to applaud the teacher who initially rejects being asked to explain her thinking in math or squawks at asking kids to do a lot of writing. They tell me that this all slows down the number work —to teach about how or why something works in math. They often say there is no point in writing why one-half divided by one-sixth is three when "it just is!" We now have a clear response to that rejection. One of the new Standards for Mathematical Practices from the Common Core State Standards (which was introduced in 2012) boldly affirms the need for mathematical argument. Therefore, I suggest you engage your teachers in active argument, share different views of problems you give them, and debate relative correct answers until together you come to the point where, no matter the method used, if you are doing math well, you will all come to the same answer. The processes used will be different and mistakes will likely be made along the way, but different processes all can lead to the same answer if the math process is accurate at every step of the way. That requires communication and debate. See the Common Core State Standards (2010), Standards for Mathematical Practices list:

(1) Make sense of problems and persevere in solving them.
(2) Reason abstractly and quantitatively.
(3) Construct viable arguments and critique the reasoning of others.
(4) Model with mathematics.
(5) Use appropriate tools strategically.
(6) Attend to precision.
(7) Look for and make use of structure.
(8) Look for and express regularity in repeated reasoning.

Touchstone Strategy #5: Journaling

Monitor everyone's learning through a math journal linked to readings in the course.

By asking all of my students (in approximately 130 courses taught) to keep a math journal, I have come to realize how invaluable the journal is both to me as teacher and to students. New teachers inform me about: 1) how they came to define math and 2) the pedagogical approaches their teachers (nursery —high school) used to teach them math. With that information, I feel I have a better awareness about how to meet their individual needs as learners. Though I see my students only once a week, I have an ongoing connection with them through their journals.

I listen hard for emotional breakthroughs in each entry and support and honor their writing, showing respect and giving assurances so they hear humanity from a math teacher. I am aware that I may be the first math teacher who has responded to their process work with human emotion. (This is what my students have told me many times!) In their journals, new teachers vent and decry the painful memories of teachers who hurt them. I respond with appreciation of their pain, but I don't stop there. I honor their expressed feelings, but also offer them new stretching experiences as well. Their writing often sounds like they are yelling at their former teachers for never exposing them to relationship thinking in math class. I let them express disheartened feelings for those teachers who forced them to copy the teacher's or the book's process. I try to point out that those teachers were not necessarily cruel. They only knew what they themselves had been taught. We teach only what we know and only what we ourselves can do. In the face of anger, I try to bring some sense of empathy for their former teachers who are often portrayed as ogres.

I offer them alternative, positive math learning experiences. As my students solve problems, they will have fellow learners sitting with them in groups and at tables who may help them write and express their new process of thinking about math. We play with math manipulatives, I offer them structured games, I invite collaboration through problem-solving groups. I work hard to assure people that expression of their anger towards their old teachers can help them heal over time (Melnick, 1992). That may take a while, but some good will come of it if they allow themselves to enjoy the learning process now. (I have to admit that some participants use the bulk of their journal to write about their own catharsis and nothing much about learning to teach. This may sound counterproductive, but if that is what they need to rid themselves of the pain inflicted on them by their elementary math teachers, the journal allows them the space to do it.)

While participating in a class built on progressive, constructivist practices in math, my students find themselves newly free to make mistakes. They tell me what it feels like to learn in a safe environment, perhaps for the first time. That is a norm that our Bank Street Math for Teachers classes are built upon. It must be so, if you really aim for emotional changes to take place. Every one of these teachers will be influencing young children's mathematical self-perceptions for years to come. Their influence on children each year will be immense. I invite them to change their views, rather than maintain negative feelings regarding mathematics teaching and learning. The regular math journaling process in this course is a venue for reflecting about what they think and what they feel while attending Math for Teachers (with feedback continually given by the teacher and assistant teacher). The journal can provide the glue to hold people "to the fire" of change. The math journal can help people reframe their views about what math teaching and learning can be. I find asking teachers to keep a math learning journal can serve as a potent change impetus. It can be useful to anyone at any age and at any time in their professional development.

On moving from being "not smart" to a valued thinker:

> In my first journal for this class, I wrote that I had classified myself as "not smart" regarding math because I felt I was unable to grasp the concepts that we were learning in math class. I wrote that this feeling followed me each year, and before I even entered into a new math class, I already had that feeling of inevitable failure. One of the first times I actually felt smart in a math class was while I was explaining to my group this semester how I conceptualized our perplexing problem. I felt that what I was saying made sense and was not "stupid," and I felt my thoughts were valuable. I realized one aspect that contributed to this positive experience was that our professors did not make us feel that we were in a time-crunch and that we could take as much time as we needed. I also realized the importance of creating a classroom atmosphere where students feel comfortable with their ideas and supported by their classmates and professors. Another aspect of this experience was that I was able to use materials to explain myself —in this case, colored blocks. Using materials allowed me to visualize what was going on in my mind. —Katherine

On the power of the journaling process:

> I remember writing my first journal for Math for Teachers in January. As I was writing it, I was thinking "do I really need to take this course as a requirement?! How will it help me?" As it turned out, EDUC 540 was memorable not only because it challenged

me, but because I realized a lot about teaching along the way, as well. In your teacher comments regarding my first journal, there was a comment about the fun in our classroom over the course of the semester dissipating. It was explained to me to not give up when it gets hard and to stick with it. When Hal wrote this to me, I was apprehensive that the class may get a lot harder and challenge me. Over the next few months, the class did surely change from the beginning. We started off with just math games and, over time, the class changed into some lectures, teamwork, and other different types of challenges. —Erica

On connecting math to joy:

My second goal—to not be afraid of math, and to not be overly concerned with being right—was thankfully handled within my comfort zone of stop-motion animation. Feeling stripped of any creativity, I do not think I would have felt at ease enough to even begin a mathematical career. The tactical tools, the number talks, and the inclusive articles, all put my nerves at bay. Sitting in a group with my dear friend Pam as well! Our minds work so differently, and both of us were curious and delighted at how the other thinks without having to justify our thoughts. That was beautiful and lucky for me. Having the tools, the teachers, the friends, and the curiosity made this class an honest to goodness life changer. I would now consider math a skill I own. I would love to continue to animate math lessons and deepen my understanding of math. Math is a lesson that is everywhere, within every meal, every baking project, and every art project, I never knew I was instinctively mathematical. Thank you. I thank you. —Kirsten

On how math doesn't have to be stressful:

First of all, this class allowed me to experience for the first time in my life that math does not have to be stressful. Even if presented with a challenging problem, that does not mean it has to be an experience filled with anxiety. The words: "math problem" can actually be associated with positive feelings! —Katherine

On getting teary eyed at the end of the course:

After class on April 25, I felt very inspired. When Hal put the long division problem on the board (47,387 divided by 7), my first thought was "Oh dear lord." But then I remembered—we've worked with numbers like this before. In fact, we've been playing with all these numbers all semester! Suddenly, I found the 7s that I could identify. I counted down and got my answer. Whoa! In just a few short months, by playing games and making math accessible for all, I was able to take apart a huge number that I would have immediately dismissed otherwise. This made me a little teary—I'm a mathematician! What?!?!?
—Kelley

Students come to own their learning by cementing their feelings when writing in their journals. We hear an impassioned sense of ownership emerging in the voices of teachers learning in a mathematics methods course built upon constructivist teaching practice (Melnick, 1992). As a student named Eileen, who's quoted in my dissertation, said,

> What you feel after each week in class is different. You don't remember things if you don't write it down. It was not only because you (the professor) wanted us to write, but because I wanted to remember the feeling. I mean this journal was going to be mine and it was going to help me and I wanted to remember . . . more than I was concerned about whether you liked it or not. It was here to help me!!!

The math journal has the potential to weave the entire course together. Through writing, people usually connect their experience to their emotional response of that experience. In math class, the amygdala (the fight or flight part of the brain) is regularly activated. Students are then expected to connect their feelings to the intellectual (class readings, the lectures, videos watched, articles discussed in groups, etc.) If they do not make those emotional/ intellectual connections, it is my job to ask the next intellectually challenging question that pushes the student past where she or he might be into a new place of understanding. If ready, they will bite. If not ready, they will argue. Good! The Common Core Math Standard # 3: "Construct viable arguments and critique the reasoning of others" will have been supremely activated. You will have done your job well.

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Three Area Measurement Tasks Connecting Area-Units, Multiplication, and Area Formula

Sayonita Ghosh Hajra and Brianna Davis

Abstract. In this article, we present three area measurement tasks, each focusing on area-unit, the meaning of multiplication, and the area formula of a rectangular region, and discuss how these tasks can support prospective teachers' thinking about area of a rectangular region.

1. Introduction

Area is one of the foundational topics in the mathematics curriculum from the early grades up, but like K-12 students, prospective teachers struggle with the conceptual understanding of area. In this article, we consider what area is and how it is measured from an elementary point of view. To clarify these ideas, we present three area measurement tasks on rectangular regions, focusing on area-units and the meaning of multiplication. The tasks were designed to support prospective elementary school teachers in thinking about area.

1.1. The Challenge. Various studies have shown that U.S. children often have trouble understanding area (Battista & Clements, 1996; Clements et al., 2018; Kamii & Kysh, 2006) and their future teachers do too (Browning et al., 2014). Prospective teachers tend to use area formulas without understanding them (Baturo & Nason, 1996; Livy et al., 2012; Simon & Blume, 1994; Tierney et al., 1990; Wickstrom et al., 2017). Often, the first response to the question, "What is area?," is—"Area is length times width." All of geometry, and area in particular, may be covered cursorily in K-12 school curricula. Procedures can be presented with little or no attention to a deeper understanding of area and area-units (Hong & Runnalls, 2020). Many elementary school textbooks reflect this, providing few lessons on area-units, arrays, and connecting area-units with area formulas (Hong et al., 2018). Prospective teachers, reflecting their own schooling, may mechanically multiply the length by the width to generate a number and be challenged if asked to explain why this area formula works conceptually and what the product means.

1.2. What is Area? Area is the amount of two-dimensional (2D) space taken up by a 2D figure. It is measured by comparing the 2D figure with another two-dimensional shape, called the area-unit. The amount of space is then quantified by the number of area-units that can cover the shape without gaps or overlaps. Any 2D shape can be used as an area-unit (see Figure 1). There are two basic principles of area: 1) Moving Principle - Area is preserved when moved, and 2) Additivity Principle - When a 2D shape is cut into pieces, the area of the shape is the sum of the

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area of the individual pieces. These two principles are crucial when computing areas using the idea of covering with area-units (Figure 2). Usually, this idea of covering with standard square units is introduced in the lower grade levels. Eventually, the counting of area-units is replaced by area formulas for quick computations in the upper-grade levels.

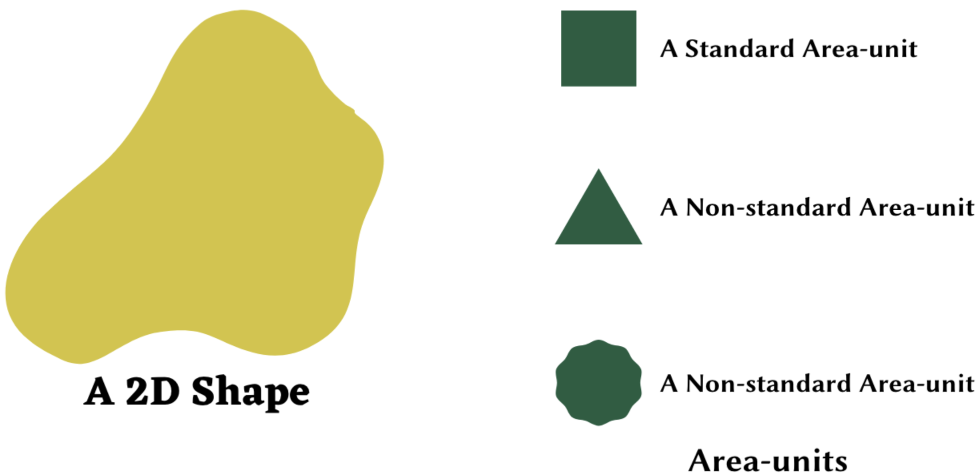


Figure 1. Examples of three different area-units.

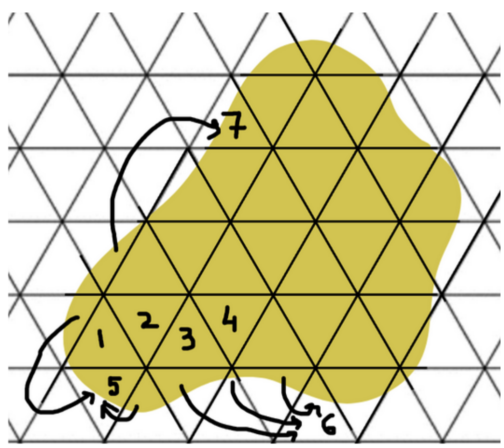


Figure 2. A triangular unit is used to estimate the area of a 2D blob by counting the number of units needed to cover the blob. Some whole units are labeled (1, 2, 3, and 4). Units labeled 5, 6, and 7 are filed by moving and adding partial pieces of other units, shown using arrows.

1.3. Why Area? Area can be a powerful visual tool that aids mathematical learning. When used as a model, area serves as a tool to visualize numbers and operations. Students can draw on the area models to make sense of the distributive property and fraction multiplication, properties that students utilize throughout academia and beyond classrooms. For example, students can use the area of a rectangular region to reason why $(5+3)\times(10+4)$ is equivalent to $(5\times 10)+(5\times 4)+(3\times 10)+(3\times 4)$ (Figure 3a) and why $\frac{2}{3}\times\frac{4}{6}=\frac{8}{18}=\frac{4}{9}$ (Figure 3b). Integrating abstract concepts in numbers and operations into visual representations (such as area models) supports student understanding of these concepts.

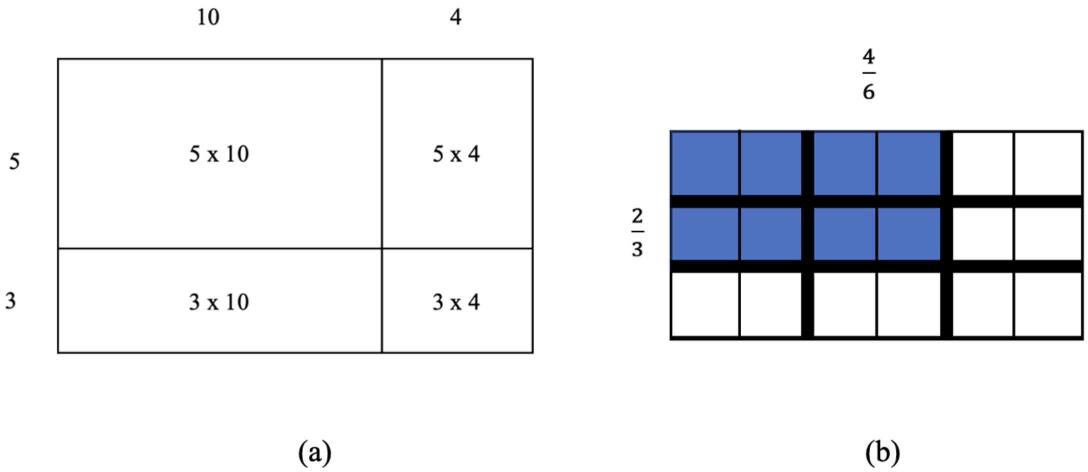
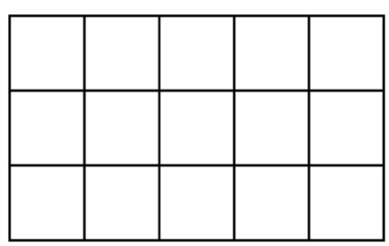


Figure 3. a) Use of area model in making sense of the distributive property. b) Area model illustrating fraction multiplication $\frac{2}{3} \times \frac{4}{6}$.

2. Covering and Counting with Area-Units

One of the first steps toward learning area conceptually is understanding the basic idea of covering a region with area-units without any gaps or overlaps and counting the number of area-units required. The following task, Task 1, is intended to support prospective teachers in developing their understanding of the covering aspect of area with an area-unit.

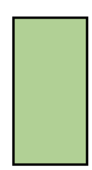
Task 1: Find the area of the rectangular region using the following 2D shapes as area-unit. Explain your thought processes.



A Rectangular Region

a) Pink area-unit

b) Green area-unit



Understanding area conceptually involves visualizing the area-unit. In Task 1, prospective teachers use area-units to cover the given region and estimate the area in terms of the given area-unit. One of the challenges we have seen is that some people cover and count the area-units correctly, but report the area with the numerical value together with incorrect area-unit such as square area-unit or $unit^2$. Hence, mathematics teacher educators must ask prospective teachers explicitly to write the area-unit clearly along with the numerical value and ask what that numerical value means. Using this task as a pre-assessment provides a good sense of someone's prior understanding of area and area-units. This task also supports developing an understanding of decomposing units. For example, two of the smaller rectangles may be used to create a bigger area-unit.

3. Array Structure and Multiplication

The next steps toward learning area conceptually are to understand covering a rectangular region with area-units systematically by forming an array structure and then using the concept of multiplication to estimate the number of area-units. Here, the multiplication "$A \times B$" represents the total number of objects resulting when gathering A groups (multiplier) of size B objects (multiplicand).

Task 2 was designed to support prospective teachers in visualizing area-units in arrays and then using multiplication to estimate the area of the rectangular region in terms of the area-units.

Task 2: Calculate the area of the blue rectangle of dimensions 31" by 13" using the orange tiles of dimensions 3" by 5". Explain your thought process and show how you calculated the area of the blue rectangle.

This task is multi-layered. First one needs to understand the area-unit used for the task is the orange tile. Then one can arrange the orange tiles in an array and use fractional knowledge to estimate the area of the entire blue region, or use multiplication to estimate the number of area-units.

Consider Figure 4 (next page), which shows 4 groups (rows) with 6 orange tiles in each group that almost cover the blue rectangular region. When the tiles are arranged as in Figure 4, with no gaps or overlaps, there is some blue rectangular space left over that cannot be covered with full orange tiles.

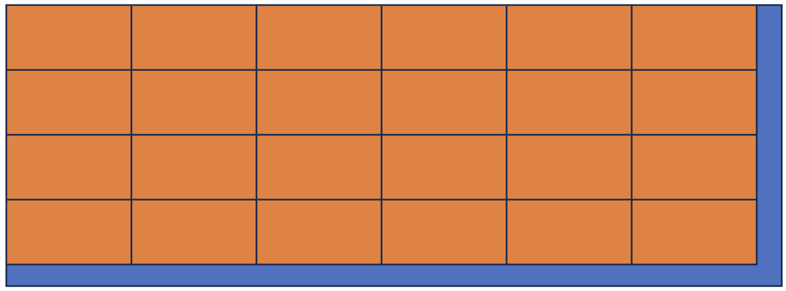


Figure 4. A 4×6 array of orange tiles covering most of the blue rectangle.

Next, one needs to estimate the number of tiles that need to be laid over that remaining region. To estimate that, one needs to understand that each of the rectangular orange tiles can be covered with square area-units (i.e., that the larger orange tile can be thought of as a 3×5 orange tile that has 3 groups with 5 squares in each group (see Figure 5).

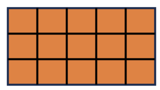


Figure 5. A single 3" by 5" orange tile covered by 1" by 1" squares: 3 groups (rows) of 5 squares in each group.

Looking at the remaining horizontal row, one can fit six one-thirds of an orange tile, four one-fifths of an orange tile, and $\frac{1}{15}$ of an orange tile (see Figure 6). Using the fractional parts of the orange tiles, the area of the blue rectangular region is $24 + 6 \times \frac{1}{3} + 4 \times \frac{1}{5} + \frac{1}{15} = 26\frac{13}{15}$ orange tiles.

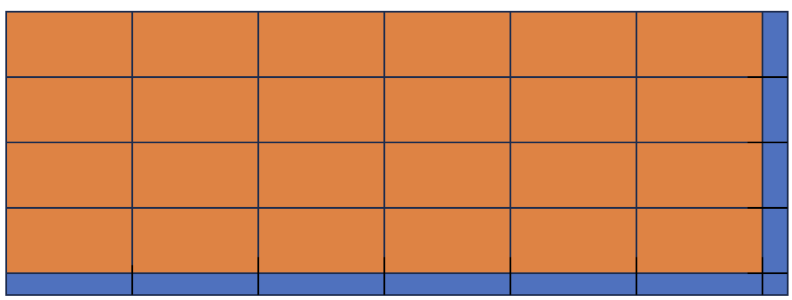


Figure 6. A $4\frac{1}{3} \times 6\frac{1}{5}$ array of tiles.

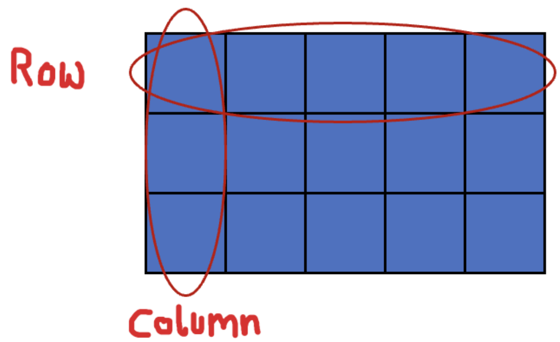
Another way to solve this task is to notice the array structure and use multiplication. There are $4\frac{1}{3}$ rows (groups) with $6\frac{1}{5}$ tiles in each row (see Figure 6), hence $4\frac{1}{3} \times 6\frac{1}{5} = 26\frac{13}{15}$ orange tiles.

This task builds on understanding of area-unit and illustrates a systematic way to count units using the concept of multiplication. Prospective teachers must be given opportunities to explore how children progress from one-to-one counting in no specific order, to systematic counting using multiplication. Tasks that surface a prospective teacher's thinking and strategies can provide information to mathematics teacher educators for structuring lessons in the classroom that can appropriately address prospective teachers' learning needs on area-related tasks.

4. Area-unit, Multiplication, and Length Times Width Formula

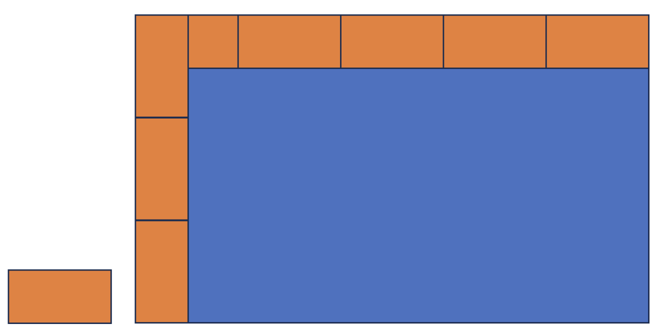
The last step is to make the connection between area-units, the concept of multiplication, and the "length times width" formula. Task 3a highlights an understanding of the connection of multiplication with the formula.

Task 3a: Rukshana explains the area of the rectangular region is 3×5 as "there are 3 rows and 5 columns" as shown in the picture below. Do you agree or disagree with Rukshana? Why?



Comprehending that the length refers to the rows (or columns) and the width refers to the squares in each row (or column) is a critical step toward understanding the area formula of a rectangular region. In the U.S., the first slot in multiplication refers to the number of groups (multiplier) and the second slot refers to the number of objects in each group (multiplicand). There are two choices for the group, a row or a column. Once a group choice is made, then the second slot is about counting how many squares are in each group. Task 3b allows prospective teachers to reconcile their understanding of using the lengths and widths with area-units construction in estimating the area of a rectangular region.

Task 3b: Mindy used the orange tiles as shown in the figure below to cover the blue rectangular region horizontally along the top of the rectangle and vertically along the left side. Mindy counts 5 across the top and 3 across the left side and says, "Area is length times width, which is 3×5."



a) How would you respond to Mindy?
b) What does Mindy know? What does Mindy need to know?
c) Using Mindy's work, how would you find the area of the blue rectangular region?

Both Task 3a and Task 3b offer a deeper look at how the length and width measurements of a rectangle using ruler generated one-dimensional edge measures is incompatible with area-units. In particular, the overlap in the upper left corner of Mindy's drawing is not a single-layer covering, Figure 7.

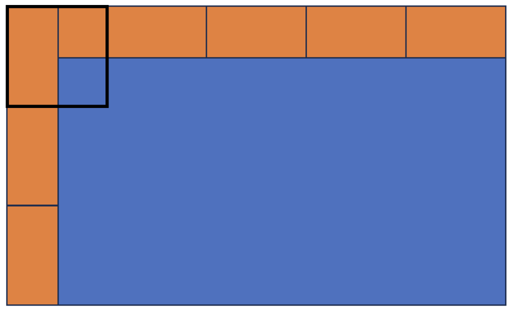


Figure 7. Incompatibility of single-layer area-unit approach and use of the length × width formula.

5. Mathematics Teacher Educators' Opportunities for Learning from the Three Tasks

Using these tasks as pre-assessment, instructors can get a good sense of prospective teachers' prior understanding of area and area-units. We used several of these tasks (1, 2, and 3a) as an assessment tool in one of the mathematics content courses for future elementary teachers. This gave us important insights into the initial understanding that students' brought with them to the class. In Task 1, most students used a covering idea. They saw that the area-unit is made up of two of the smaller units (Figure 8).

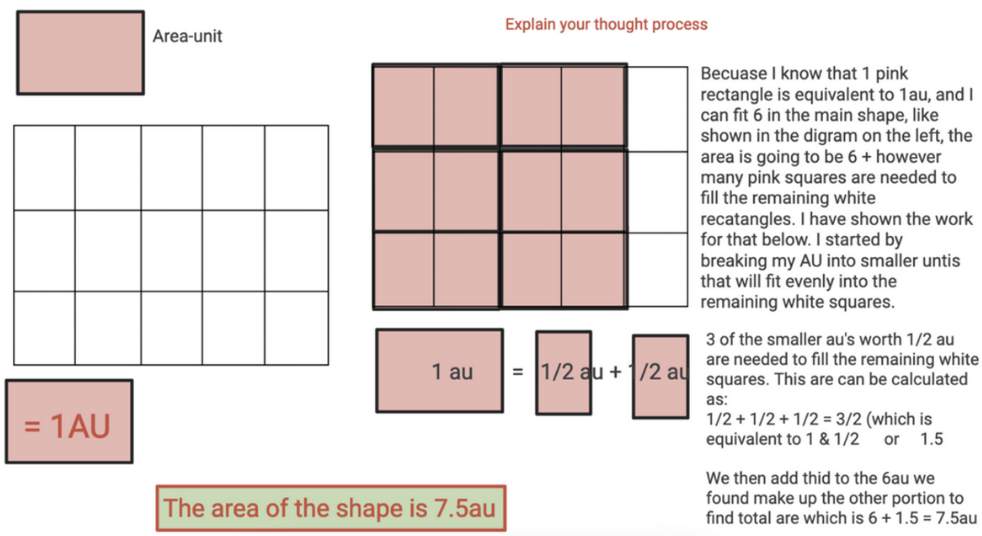


Figure 8. A prospective teacher's work on Task 1.

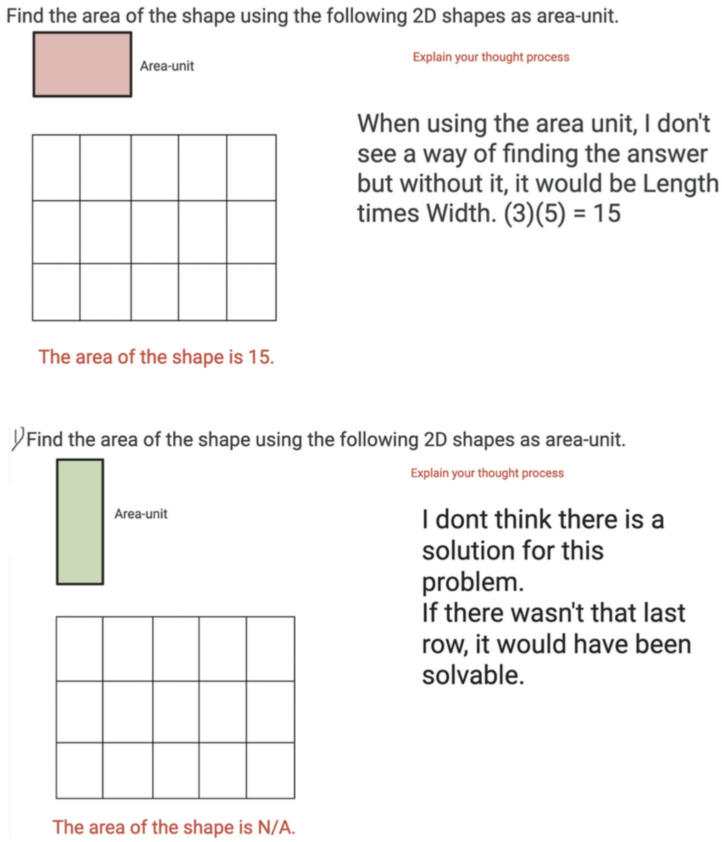


Figure 9. A second prospective teacher's work on Task 1.

We also found that a few had some difficulty conceptualizing area-units. In Figure 9, a prospective teacher used the area formula, length times width, for Task 1 and did not see the area-unit as made up of two smaller units.

Task 2 provides prospective teachers opportunities to work with whole and fractional area-units. On their first attempt, most looked for a way to use only whole orange tiles to cover the blue rectangular region. Doing so, two realized that there were areas left out as the whole orange tiles do not completely cover the blue rectangular region (Figure 10, next page).

The other prospective teachers started with covering, but then shifted away from the visual tools to using the area formula. For example, one prospective teacher wrote:

> Finding the area using the orange square was going to be more difficult because the squares [orange tiles] do not cover the whole unit [blue rectangular region]. There is a small amount of space between the base and the height that it left open that we would need to add on top of our calculations. The easiest way would be to calculate the area of the blue rectangle itself to find the area the easier way A = L×W = A = 31 × 13 = 403.

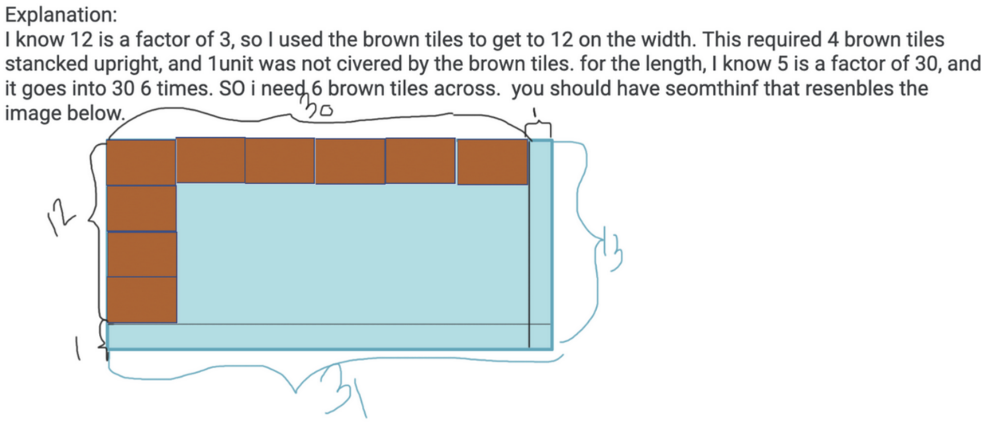


Figure 10. A prospective teacher used orange tiles to line up along the boundaries in an L-shape.

Another prospective teacher wrote:

> Since they wanted us to calculate the area of the blue rectangle of dimensions 31" by 13" using the orange tiles of dimensions 3" by 5", I noticed that it took 20 of the blue [sic] [orange] to fill in the entire orange [sic] [blue] rectangle. I don't think that it mattered because I still had to multiply 13 by 31 in order to find the area of the orange [sic] [blue]. The area I found for the orange [sic] [blue] rectangle is 403.

A third prospective teacher wrote:

> Finding the area of the blue using the orange; what I did first was look at the number dimensions. With the 5 and 31; with the blue having 31, I know it could have at least 6 orange tiles go across since 5*6 is 30 and 30 is less than 31. This means there is 1 inch left over on the side. With the 13 and 3, I knew there could be at least 4 orange tiles going up and down leaving 1 inch off because 3*4 is 12 and the goal is 13 on the blue rectangle. With the orange tiles in the blue rectangle it covers 30 inches by 12 inches, (30*12=260 [sic] [360] inches) that leaves 1 inch uncovered by orange tiles on both the width and the length of the blue rectangle, so 13*31 = 403 inches.

In all of this work by prospective teachers, it was evident that the area-units did not play a strong role in the area calculation in their initial thoughts. Looking at the reported units, we see that two of the responses did not have a unit and the third one reported the unit in "inches."

We found that our prospective teachers came up with a strong idea of covering and counting units when the area-units would completely cover the region to be measured. Struggles were observed when area-units needed to be decomposed into smaller units in order to cover completely a given region. Also, we observed the tendency of the prospective teachers to apply the length $\times$ width formula irrespective of what the problem asked.

Identifying prospective teachers' initial difficulties provided opportunities to have a conversation in the classroom. In the next attempt at Task 2, prospective teachers were invited to work in groups. They were given index cards and a 13" by 31" rectangular paper region (Figure 11). They were also instructed that they were allowed to tear the index cards as needed to fill the rectangular region. The final work is shown in Figure 12.

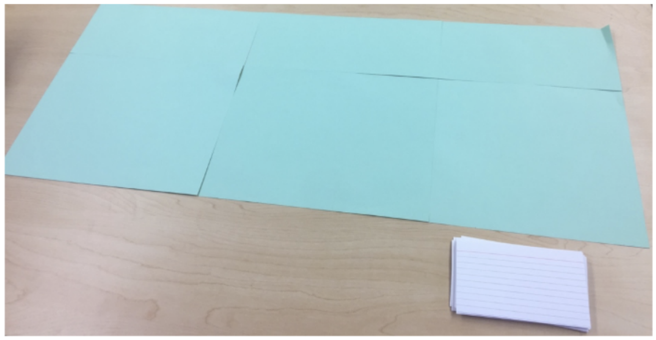


Figure 11. Index cards and 13" by 31" rectangular paper of region for a second try at Task 2.

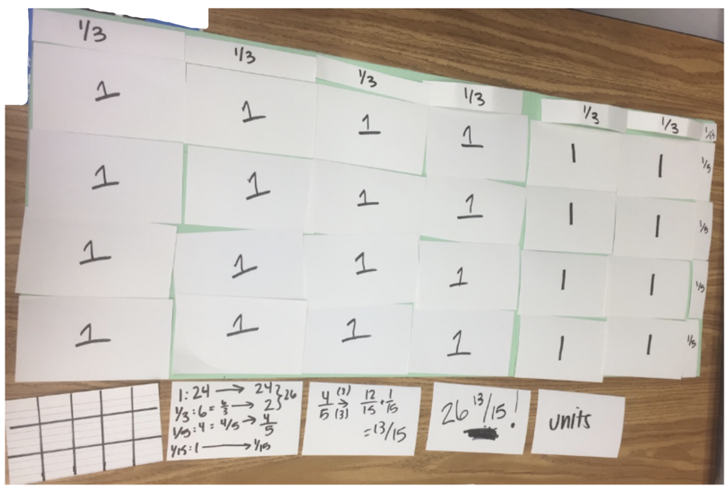
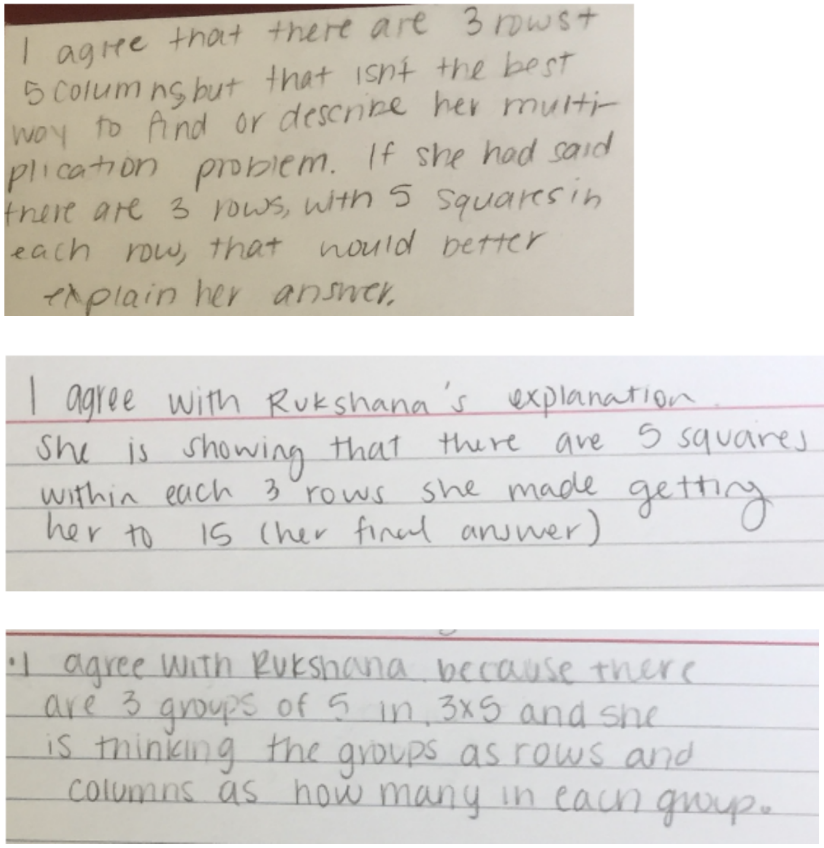


Figure 12. Work by a group of prospective teachers on Task 2, after class discussion.

Prospective teachers covered the rectangular region with whole and partial area-units and counted the number of index cards. Finally, this arrangement of cards in an array supported a class discussion of how to visualize this as a multiplication problem (i.e., $4\frac{1}{3}$ rows (groups) with $6\frac{1}{5}$ tiles in each row).

In Task 3a, initially only one prospective teacher disagreed with Rukshana. This prospective teacher wrote, “I disagree. The area is 3×5 because there are 3 groups of 5 or three rows of 5.” For this prospective teacher, the use of multiplication was clear. There were three prospective teachers who used the common U.S. language around multiplication: that the first slot refers to the groups and the second slot refers to the number of objects in each group. However, they did not articulate what the group and the objects in each group meant (Figure 13).

Two other prospective teachers agreed with Rukshana’s solution but did not communicate an understanding of multiplication (Figure 14).

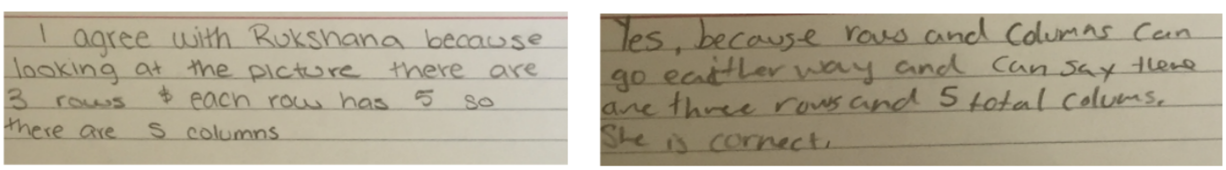


I agree that there are 3 rows + 5 columns but that isn't the best way to find or describe her multiplication problem. If she had said there are 3 rows, with 5 squares in each row, that would better explain her answer.

I agree with Rukshana's explanation. She is showing that there are 5 squares within each 3 rows she made getting her to 15 (her final answer)

• I agree with Rukshana because there are 3 groups of 5 in 3x5 and she is thinking the groups as rows and columns as how many in each group.

Figure 13. Prospective teachers agreed with Rukshana's solution, but did not fully give specifics for groups and objects in each group.



I agree with Rukshana because looking at the picture there are 3 rows & each row has 5 so there are 5 columns

Yes, because rows and columns can go eaither way and can say there are three rows and 5 total colums. She is correct.

Figure 14. Responses to Task 3a showing prospective teachers agreeing with Rukshana's solution though offering incomplete explanation.

Task 3a helped prospective teachers to understand the role of identifying the groups as rows or columns in an array and an area-unit as an object. After class discussion, prospective teachers revisited their initial responses and reevaluated their understanding (Figure 15).

6. Summary

A valuable step in learning area measurement is to develop a good understanding of area-units. Once the appropriate area-unit is chosen, the next step involves a comparison of the chosen units with the given shape. The last step in the area measurement is systematically counting the

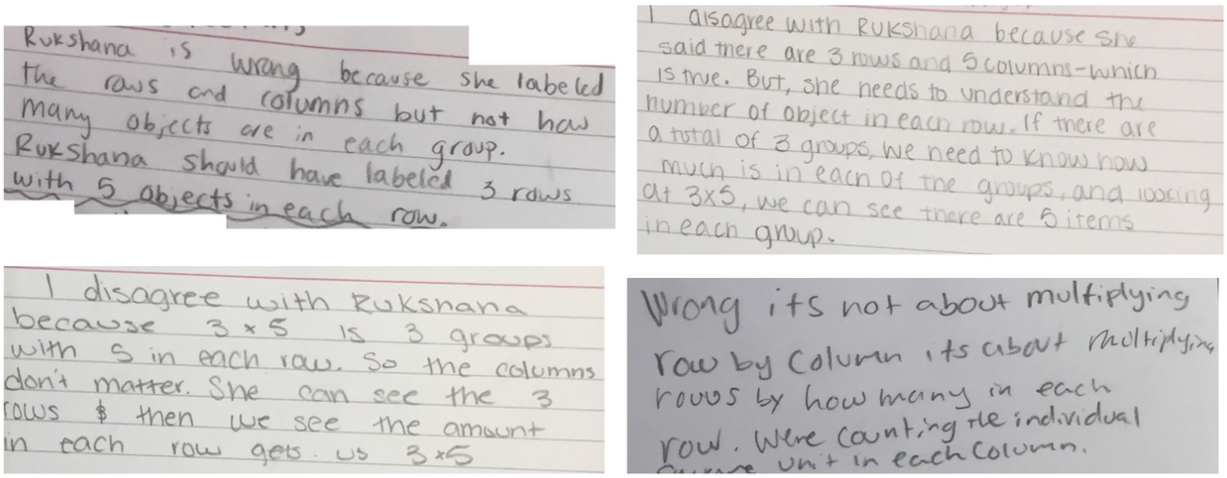


Figure 15. Prospective teachers' responses to Task 3a after class discussion.

area-units. This counting could involve counting individual units, counting by grouping, or counting using sophisticated multiplicative groupings. Specifically, visualizing the area of a rectangular region as an array (rows or columns) of square or non-square units is one of the fundamentals in calculating the area of a rectangular region (Simon & Blume, 1994).

The main struggle points for prospective teachers are often about reporting area using the appropriate area-unit and differentiating strategies for when or when not to use area formulas to calculate the area of a rectangular region (Baturo & Nason, 1996; Menon, 1998; Reinke, 1997). As mathematics teacher educators, we can leverage the tangible and visual aspects of covering to introduce area with a focus on identifying the area-unit when measuring the area of a 2D shape or a rectangular region.

Providing prospective teachers opportunities to play with the basic idea of covering a region with tangible area-units and asking explicitly about area-units in reporting their answers could support their understanding of the meaning of area. Appropriate tasks in the content courses can assist prospective teachers in revising their thinking and sense-making of area and relating the meaning of multiplication with the area formula of a rectangular region, (i.e, connecting the length with the number of groups and the width with the number of squares in each group).

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MDTP as a Formative Tool to Support Students' Access to Mathematics: Identify, Analyze, and Enact

Kimberly Samaniego and John Sarli

Abstract. Mathematics Diagnostic Testing Project (MDTP) is a grant-funded effort supported by the University of California and California State University (UC/CSU) to provide a free assessment system with professional training to California mathematics educators from secondary and postsecondary institutions. When used formatively, MDTP diagnostic tests and online testing platforms help teachers to identify students' strengths, unfinished learning, and potential gaps of content knowledge. This data, when used in accordance with the MDTP Formative Assessment Framework (FAF), can inform the design of instructional actions to support students' access to mathematical content. In this article, we summarize the history of MDTP, the test-development process, and the components of the MDTP assessment system. Then, we share the components of the MDTP FAF and examples of how MDTP diagnostic results have been used to inform learning experiences in a post-secondary, entry-level mathematics bridging course.

1. Introduction

Formative assessment is a valuable tool for addressing post-pandemic instructional challenges. Mathematical content that students missed or have not yet mastered may need **bridging** in order to promote access to new content. It is critical for mathematics faculty to have access to what our students know when starting a new topic or unit so that we can design and enact appropriate instruction. Implementing an effective formative framework supports the process to learn what students know, determine what students need to know, and then design and enact instruction to bridge between them.

The CSU/UC Mathematics Diagnostic Testing Project (MDTP) offers diagnostic tests and online testing platforms as tools to help teachers quickly identify students' strengths, unfinished learning, and opportunities for bridging content knowledge. Ideally, data received after administering an appropriate MDTP grade-level assessment of preparedness or course-level readiness is used in accordance with the MDTP Formative Assessment Framework (FAF), which emphasizes the following activities.

- **Identify:** Identify students' current mathematical understandings, misconceptions, and potential gaps in content knowledge.
- **Analyze:** Unpack the progression of mathematics that students need to build the essential understandings needed for access to and mastery of the content.
- **Enact:** Adopt strategies and design learning experiences to support these learning goals.

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In this article, we describe how MDTP diagnostic results and the MDTP FAF were used in a post-secondary, entry-level mathematics bridging course to precalculus. These efforts informed the learning experiences for a remote/online course.

2. Background and Overview

MDTP was jointly formed and funded by the California State University (CSU) and the University of California (UC) in 1977 to promote and support student readiness and success in college mathematics courses. The California Education Code (CED) Title 1 established the California Academic Partnership Program (CAPP) in 1983 in order to develop cooperative efforts for improving the academic quality of public secondary schools, with the objective of preparing all students for college. Since its inception in 1989, under Article 3.5 of the CED, MDTP has been developed to provide materials and outreach **free of charge** to California mathematics educators from secondary schools and the UC and CSU college systems. CAPP provides oversight for the work and assignment of MDTP site directors on designated UC and CSU campuses. The diagnostic purpose of the MDTP guides the development of all MDTP materials which are aligned to the Mathematics Framework for California public schools.

MDTP tests are developed by MDTP Workgroup members. The Workgroup is the governing body of MDTP and includes faculty from CSU, UC, California Community Colleges, and secondary schools and districts. The Workgroup develops diagnostic assessments and open-response items that measure students' mathematical preparation in foundational topics essential for success in the course students are entering. CSU/UC MDTP assessments are copyrighted, and their content may not be used in other test forms.

MDTP provides outreach and support to secondary mathematics educators to interpret and use test results formatively via site visitations, conferences, and workshops. Representatives of MDTP play significant roles in supporting mathematics preparation through collaborative work with educational agencies across the state. Examples of these efforts include decades of providing feedback on each update to the *Mathematics Framework for California Public Schools*, participation on advisory boards and committees for the Center for the Advancement of Instruction in Quantitative Reasoning, California Mathematics Education Collaborative, Just Equations: Advancing a Mathematics of Opportunity, and the California Mathematics Project. Additionally, MDTP works with many organizations and programs that support students from underrepresented populations through sponsored projects and summer programs connected to CSU and UC campuses.

3. Reliability and Validity

An essential element of the test-development process is an extensive statistical review to ensure that each item assesses intended target knowledge and is reasonably predictive for students from all skill levels. Measures of reliability and validity stem from this process.

Since **reliability** measures consistency of data patterns over repeated administrations of a test, the MDTP Workgroup authorizes a released form only after analyzing a sequence of field-test forms, typically a process that takes several years. The primary measure of reliability that MDTP uses is the Kuder-Richardson Formula 20 (KR-20), which measures the ability of a measure to distinguish between students who are well prepared in the tested subject material and those who are less prepared. (KR-20 values greater than 0.50 are considered good. MDTP values are always at least 0.80 with many in the 0.90 and above range.) Focused attention in the development process is given and reflected in the item quintile graphs that compare quintile item scores with overall quintile scores on the test. All items on released MDTP test forms have increasing graphs, most with slopes greater than 1.

Unlike standardized tests which are norm referenced, MDTP tests are criterion referenced. When an item is reviewed after field testing, close attention is given to how it addressed the topic specification for which it was chosen. In almost all cases, the Workgroup requires the mean test score of each group of students who chose a particular incorrect answer option to be below the total mean test score. The mean test score for those who chose the key is required to be above the total mean test score. An exception may occur for an incorrect option representing a common misconception. In this case, the group who chose this option may have a group mean score above the total mean test score, but still below the group mean test score for those who chose the key. These requirements ensure high KR-20 values and thus test reliability.

MDTP has several measures of validity. An essential first measure is **face validity**. Each MDTP test-development committee that writes and assembles a college test has university and community college members who are familiar with the college course for which the test is measuring readiness. The development committee for each middle school or high school test has, in addition to college faculty, grade and course appropriate middle school and/or high school teachers. Once a test is assembled, all MDTP members review the test to validate the content. After each test is assembled and reviewed, it is field-tested with students at the appropriate course and grade level. Measures of **course validity** are implemented to correlate field-test results with post-test measures:

- At the college level, final course grades are collected, and MDTP test scores are correlated with matching course grades.
- At the middle school and high school levels, end-of-year scores from the released form of the next-level MDTP test and teacher evaluations of student readiness for the next-level course are collected. MDTP field-test scores are correlated with both the next-level MDTP test matching scores and with the matching teacher evaluations.

Typically, these correlations are above the 60% level with many in the 80% level. All released forms of MDTP tests have been evaluated for both face and course validity, so correlations of field-test scores with released MDTP test forms provide a further measure of course validity. Detailed statistical analyses of MDTP test forms conducted in the mid-1990s have informed this outline of reliability and validity. Such a study can be found in Manaster and Gerachis (1995).

4. The Components of the MDTP Assessment System

The MDTP Assessment System provides diagnostic tests, open response Items, and MDTP Learning Modules to support students entering grade six through calculus. MDTP results provide just-in-time results to educators to inform their planning for instruction so it builds on students' current mathematical understandings. Results provide information so instructors may act on identified unfinished learning of content knowledge. For a full list of MDTP content aligned to the California State Standards for Grade 6 through Calculus, see Topic Progressions Aligned to Standards.

4.1. Diagnostic Tests. The tests include grade-level assessments of preparedness and course-level readiness tests that can be administered online or in paper booklets with paper-based scoring. Both modalities offer student results online and provide robust features to view and analyze diagnostic data to select aggregate views, sort columnar data, and drill down to item-level and student-level results. These tools benefit educators by informing the instruction needed to support students' success in mathematics coursework. Assessments of Preparedness and Readiness Tests measure students' preparedness in the foundational topics of their current or subsequent course of study. The results are intended to be used formatively to:

- Help teachers understand students' mathematical strengths and areas of unfinished learning to inform instruction and intervention
- Evaluate course readiness
- Measure program growth
- Identify content for professional learning

MDTP recommends sharing the diagnostic results with students to help them identify areas of strength and topics in which additional study or review is needed. The webpage for MDTP 9th Grade Assessments provides one reliable indication of the extent to which a student's current mathematical proficiency matches the skills and knowledge needed for success in a ninth-grade mathematics course. The results should be used as one of multiple measures to inform appropriate options for students' mathematical engagement in coursework towards college preparedness. MDTP recommends administering the appropriate course-level readiness test within the first month of the students' start in their enrolled 9th grade course, as indicated by Senate Bill 359 (California Mathematics Placement Act of 2015).

4.2. Open Response Items. Open-ended response items provide opportunities for students to show their understanding of key mathematical concepts. The Written Response Items (WRI) elicit student thinking and quantitative reasoning that aligns to MDTP topics on MDTP diagnostic assessments. Most WRIs can be administered across multiple levels of mathematics coursework. WRIs require adequate time for students to think and reason deeply about the problem and to clearly explain or justify their thinking. Formative Constructed Response Items (FCRI) are a focused assessment designed to provide a post-intervention check of students' understanding of a key mathematical topic or concept. They should be administered after students have taken an MDTP assessment of preparedness or readiness test and an intervention has occurred.

4.3. Learning Modules. The design of MDTP Learning Modules was informed by diagnostic results from MDTP assessments. The modules were created to support independent practice by students in identified MDTP topics. These student-centered modules can be used to review content before or after an assessment, prior to entering a new course, or at the direction of their mathematics instructor.

5. Using an MDTP Framework Formatively to Bridge Learning: Results from a Bridge Class

Mathematics faculty from UC San Diego worked in collaboration with MDTP to create four diagnostic tests (taken online and unproctored) that assess readiness for students entering postsecondary mathematics courses: Bridge Course, Precalculus, Calculus 1, Calculus 2. In addition to assessing course-level readiness, these four tests were designed with some items in common so that readiness could be evaluated as students made progress through their mathematics coursework.

Evaluating improvement on common items for students whose majors required at least two quarters of calculus was an important part of the assessment process. The Bridge Course was an eight-week online course covering four units: 1. evaluate and interpret functions, 2. linear relationships and functions, 3. exponential relationships and functions, and 4. quadratic relationships and functions. Here we report on the results for students in the summer Bridge Course and how those results were used formatively to support planning for Unit 2 of the course.

5.1. Test Administration. During the first week of the program, all students took the Algebra and Quantitative Reasoning Readiness (AR) test on the second day of instruction. Additionally, during week five, students took the same AR test to evaluate mid-course growth. The figures shown in this section are screen captures taken directly from the MDTP online testing platform. The online testing platforms give faculty web-based access to results with drill-down capabilities (e.g., underlined text in blue is a hyperlink to deeper analysis of the data).

5.2. Baseline Data Analysis. The first administration of an MDTP test provides baseline data that can be used to inform three levels of formative analysis:

- *Class Readiness by Topic* shows students' overall readiness in foundational topics. This can be used to inform first steps in course pacing and unit planning.
- *Item Analysis* helps identify class strengths, unfinished learning, and potential gaps of content. This can be used to inform lessons and determine the need for re-engagement and/or intervention.
- *Individual Analysis* helps identify what individual students know. This can be used to inform whole-class and differentiated learning experiences.

The topic areas of the MDTP Algebra and Quantitative Reasoning Readiness Test are:

DECM: Decimals, including Applications; Percents; Absolute Value,
EXPS: Exponents & Square Roots; Scientific Notation,
FNCT: Functions and their Graphs,
FRAC: Fractions; including Applications,
LINR: Linear Equations & Inequalities & their Graphs; Absolute Value,
POLQ: Polynomials & Quadratic Equations & Functions,
RATL: Rational Expressions

For the pre-test administration of the AR, baseline class readiness by MDTP topic is shown in Figure 1, where each bar represents the average percent correct in each topic.

Figure 1. Pre-test Algebra Readiness Class Average Topic Scores

Another view of topic-level readiness is shown in Figure 2, which identifies the percent of students performing at the Critical Level in each topic. MDTP defines the Critical Level as the minimum number of correct responses in a topic to show adequate preparation in that topic.

Students at or above Critical Level

Export as Excel

TOPIC ▲	ITEM COUNT	CRITICAL LEVEL	NO.OF STUDENTS	STUDENT PERCENT
DECM: Decimals, including Applications; Percents; Absolute Value	5	3	21	81%
EXPS: Exponents & Square Roots; Scientific Notation	6	4	11	42%
FRAC: Fractions; including Applications	5	3	12	46%
FNCT: Functions and their Graphs	4	3	12	46%
GEOM: Geometry	5	3	16	62%
LINR: Linear Equations & Inequalities & their Graphs; Absolute Values	8	6	3	12%
POLQ: Polynomials & Quadratic Equations & Functions	4	3	7	27%
RATL: Rational Expressions	3	2	8	31%

The Critical Level for each topic is what is considered to be the minimum number of correct responses for a student to show adequate preparation in that topic.

Figure 2. Pre-AR Critical Level Performance

Evident in Figure 2 is a high percentage of students meeting the critical level target in DECM (81%) and GEOM (62%). The results also indicate that students need the highest levels of support in the LINR topic. For most effective use of the diagnostic data, MDTP recommends focusing on questions in topics that will inform the design of current and upcoming instruction. For the class in the bridge program, the first and second unit focused on linear relationships and functions. Therefore, how students did on the items in the topics of LINR, FRAC, and RATL was used as a base-line analysis to determine the mathematics that **students know now** and design instruction to bridge to **what students need to know** to access the unit content.

5.3. Formative Assessment Framework (FAF) in the Bridge Course. The instructor of this bridge course used the components of the FAF, described in the Introduction, to inform the instructional decisions during course planning and classroom implementation of lessons.

5.3.1. *Identify.* In this first step of the FAF, MDTP suggests the following guidelines when reviewing and identifying results at the item level:

- **Strengths** are items that most students answered correctly.
- **Unfinished learning** is shown as a common **misconception**, where students choose an incorrect option at a high rate, or as an informed guess, where options seem to be selected at random which likely represent **incomplete processing** of content learned but not previously mastered.
- **Gaps** occur when students omit an item (intentionally skip the question) at a high rate.

Note that the recommendation is to first identify items that are strengths for most students. For example, a question in the LINR topic might show **strength** (96% correct) in translating a linear situation from words to an equation, and another question in this topic might show **strength** (88% correct) in isolating a variable given a literal linear equation.

Possible unfinished learning in this topic can be seen in the distribution of student solutions. In the first example, Figure 3, students evaluated an expression containing operations on integers with absolute values. While 42% of students responded correctly (Option A), 23% shared a

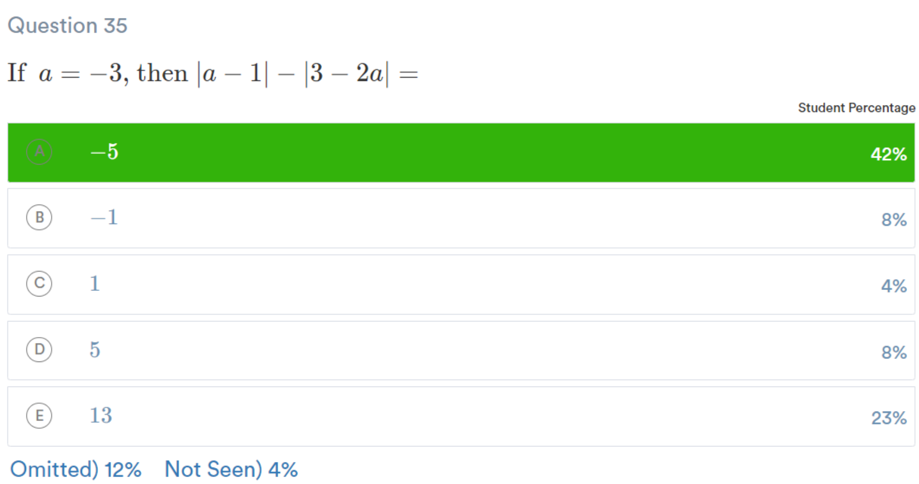

Figure 3. Question 35 from the LINR Topic.

common **misconception** (Option E), ignoring absolute value and combining $-4-9$ to get -13. Another 12% omitted the problem, while 4% did not get to the problem, and the remaining 20% combined integers erroneously in various ways (Options B, C, and D).

Question 19, in Figure 4, from the FRAC topic shows student responses for adding mixed numbers in context. Nearly one-quarter of the students omitted this problem and the same proportion of students chose Option C, a common **misconception** obtained by adding the integers $(2+1+1=4)$, the numerators $(2+5+7=14)$, and the denominators $(3+6+8=17)$ to arrive at the solution of $4\frac{14}{17}$. Overall, the high proportions for both omissions and common misconception indicate student understanding of this content likely had **gaps**.

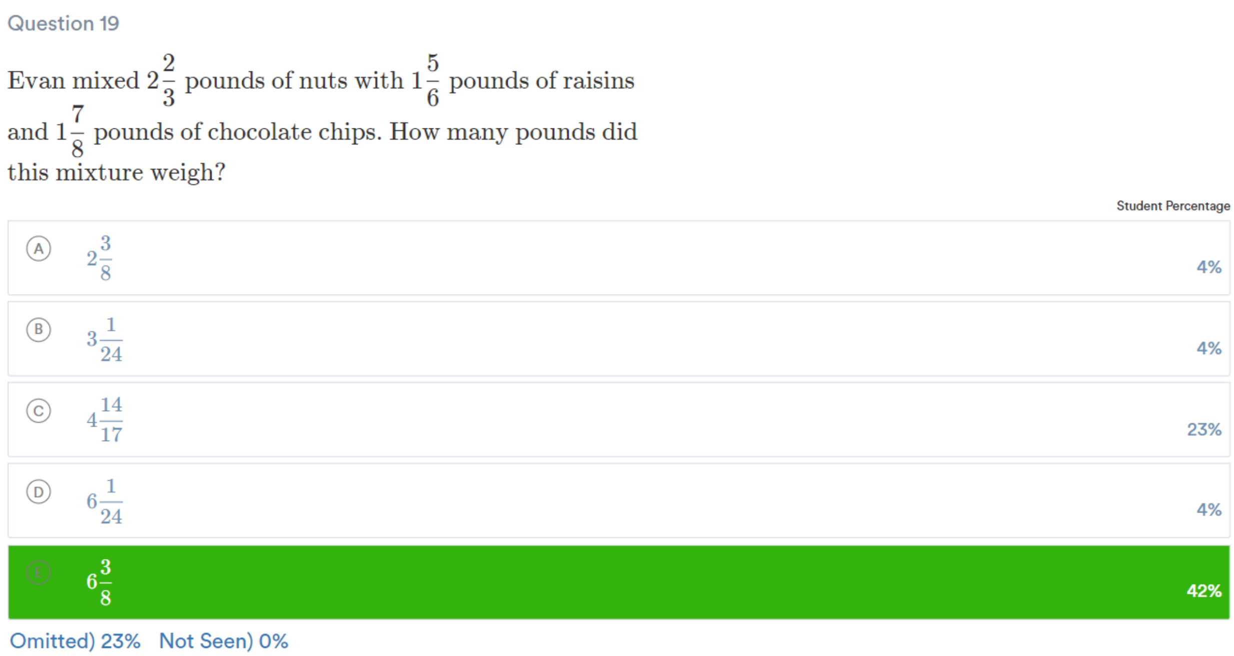

Figure 4. Question 19 from the FRAC Topic.

In Question 40, in Figure 5 (next page), from the RATL topic, students show a **misconception** when subtracting rational numbers with 42% (Option A) expressing 1 as the numerator and combining the denominators. The high rate for omission (23%) also indicates that subtracting rational expressions may be a **gap** in content for some students.

Operations on rational numbers were not included in the curriculum of the Bridge Class. Yet, left unattended, misconceptions about combining numerators and/or denominators would remain a fixed learning (conception) for students as they worked to make progress to the next level on course concepts. Thus, intervention is a vital component to bridge what the students know now to what they need to know to be successful in a precalculus course (Coleman, 2020). In summary, after full analysis of MDTP items used to inform unit planning, the instructor identified:

- **Strengths** in translating words to expressions and solving one-variable equations
- **Unfinished learning** in operations on fractions and integers, order of operations, solving systems of equations, and solving multi-step problems
- **Gaps** in solving linear inequalities and absolute value equations and inequalities

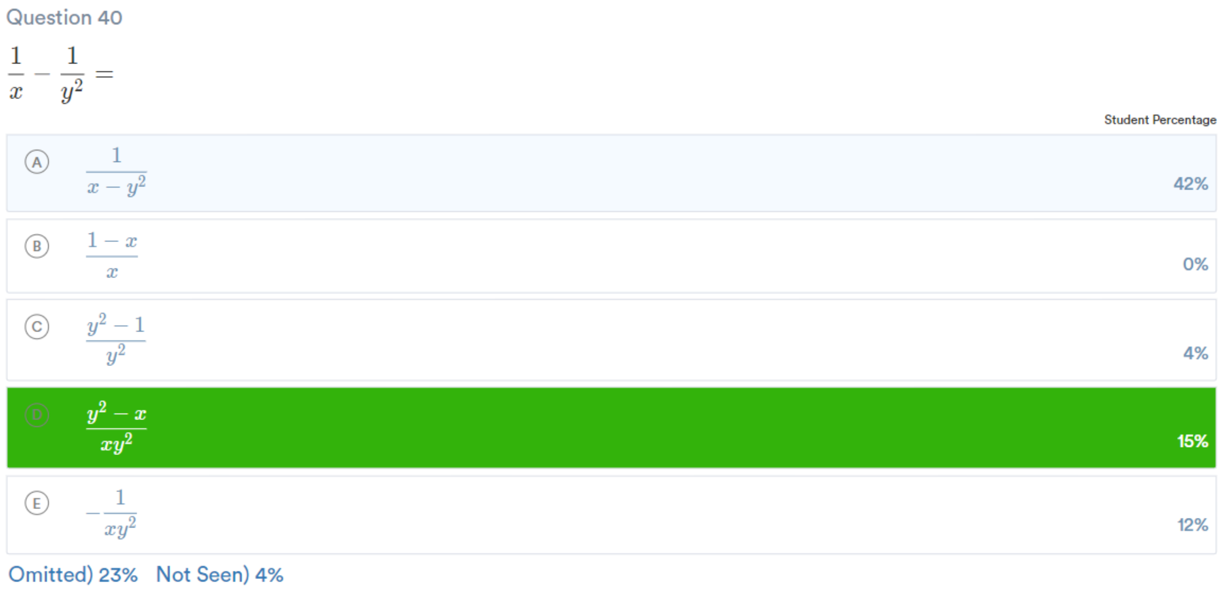

Figure 5. Question 40 from the RATL Topic.

5.3.2. *Analyze.* The next step in the FAF is to **analyze** what students know. For example, the instructor can consider the progression of mathematics building to the use of fractions in college-level mathematics. In Grade 5, students learn to add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions. In Grade 7, students apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers. Eventually, in calculus, students use partial fractions to compute the integrals of rational functions. To intervene on the unfinished learning and address gaps, students need to reengage in learning experiences that allow opportunities to revise their understanding, questioning the mathematics contained within the error and building skill with the correct mathematics both conceptually and procedurally. Attention to adding fractions in simple forms starts this process of intervention. However, leaving students at this level (Grades 5 and 7) will not provide adequate preparation for success in a precalculus course (Coleman, 2020). Therefore, extending the expectation from performing operations on simple fractions to operations on rational expressions bridges this learning expectation (see section 5.4).

5.3.3. *Enact.* The last step of the FAF is to **enact** strategies to support the identified learning goals. For this unit, the instructor optimized instructional time by dropping lower priority lessons, integrating multiple topics, and distributing the learning over time to build retention. Also, to learn more about student thinking, the instructor used MDTP Written Response Items after reviewing diagnostic data (see section 5.4.2).

5.4. Optimize Instructional Time to Integrate Topics and Build Retention. The instructor saved instructional time by dropping planned lesson time for demonstrated areas of strength. These included translating word problems involving linear relations and solving two-step one-variable linear equations. The result was increased time for solving multiple-step complicated equations with rational coefficients and signed numbers - identified as areas of **unfinished learning**. Eventually, these concepts would be applied to solving linear inequalities, a topic with demonstrated **gaps** for students. Items from other topics were useful in unit planning to integrate concepts among topics, which is essential for robust mathematical understanding. Items in FRAC and RATL were chosen to inform lesson plans involving solutions of linear equations. These topics were not specifically called out as units of study for this course, but rational expressions concepts can be woven into the topics of solving linear equations and inequalities.

5.4.1. *Re-engaging Rather than Reteaching.* The instructor created linear equations with rational solutions and then asked students to check their solutions algebraically. This strategy is an example of the necessity principle in which students learn the mathematics that we intend to teach based on intellectual need (Harel, 2008). This principle asserts that students learn concepts that are introduced with reasons that they understand and appreciate (Harel, 1998). Additionally, reteaching isolated topics of mathematics not only lacks intellectual purpose but is generally presented procedurally, with a low cognitive load, and focused specifically on students who are considered underachieving (Inside Mathematics, n.d.). Thus, reteaching may be perceived by students as devaluing the learning that has already occurred.

Additionally, students who practice solving problems incorrectly need to unlearn and relearn the associated skill(s), which is a challenging process for students (Sousa, 2017). When diagnostic data shows a common misconception, learning experiences must be designed to re-engage students in the fundamental concepts to address the misconception in short lessons that are spaced over time (Sousa, 2017, Rohrer, 2009). Mastery learning, retention, and increased metacognition through repeated active retrieval (Karpicke, 2008) are facilitated via spiral, distributed, or spaced learning with deliberate rehearsal or practice (Sousa, 2017, Karpicke & Bauernschmidt, 2011).

In this example, performing operations of addition, subtraction, multiplication, and division of signed and rational numbers was necessary to solve the problem and was a strategy used to intervene on the previously identified **misconceptions**. The following example problems were introduced in the first lesson on solving single-variable equations.

A. Start the lesson with problems that engage students in their strength. This validates the prior knowledge that students bring to the lesson.

   Problems 1 and 2: *Solve the following equations for "x" and check your answers by substituting in for x. Send private Chats to other students to check your answers.*

   1. $5x - 14 = -10 + 3x$ 2. $-3(4x + 3) = 43 - 4(6x + 1)$

B. Introduce problems with fractional coefficients. This necessitates operations with rational numbers.

   Problems 3 and 4: *In your groups, devise a strategy for solving the equations below. Document your thinking and show all your steps in your notes. Check your answers.*

   3. $12 - \left(\frac{2}{3}x + x\right) = 2$ 4. $\frac{x-5}{4} + \frac{1}{3} = \frac{5}{6}$

C. Conclude with an **exit-level problem**. This multiple-step problem necessitates multiple types of operations on signed and rational numbers.

   Problem 5: *Solve the equation algebraically. Clearly show all your steps. Check the solution in the left- and right-hand side of the equation.*

   5. $4(3x - 7) - (2x - 5) = 3(2x - 7) - 4x$

To distribute learning over time, exit-level problems presented during class were used to create three parallel benchmark quizzes that were administered over the next few weeks in class. These quizzes were designed to intervene on misconceptions, provide adequate practice (rehearsal) on identified gaps, and build retention. Problem 5 is one example of the type of question used for one benchmark quiz.

5.4.2. *Use of MDTP Written Response Items to Re-engage Learning.* In one question (see Figure 6), students are asked to determine the fraction with the greatest value among five different fractions.

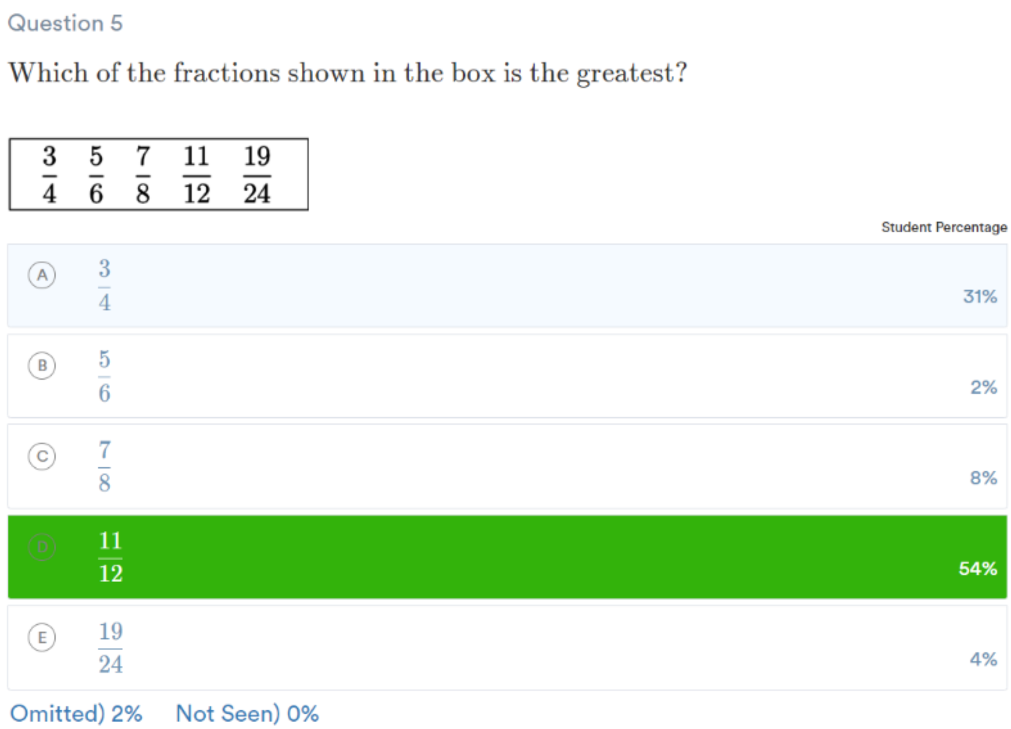

Figure 6. Question 5 from the FRAC Topic.

While 31% of the students selected the same incorrect answer, indicating a **misconception**, this question was a **strength** for the majority of the students (54%) who provided the correct solution. Therefore the instructor assigned an MDTP Written Response Item, *Fraction Order* (partial view shown in Figure 7, next page), to re-engage rather than re-teach content that was previously learned in middle school (Inside Mathematics, n.d.).

The fraction ordering material was designed to scaffold the content from Question 5 with opportunities for students to show their work and justify their reasoning. All MDTP WRIs provide a 4-point rubric to assist teachers in scoring student work. For full points, students must:

- Provide a correct answer to the question with correct justification using drawings, diagrams, mathematics or words for Parts A and B and
- Include correct comparisons and ordering with correct justification using drawings, diagrams, mathematics, or words for Part C.

Student Name ______________________ Class ______________________ Score

For a complete response: **express** your thinking in words; **label** any figures you draw; **identify** any formulas you use; **make clear** the source of any numbers you use.

A debate has started between Ronnie, Mimi, Josh, and Jill. They have been given a pair of fractions and are asked to determine which one of the two fractions has the greatest value.

A. Ronnie is convinced that $\frac{3}{4}$ is greater than $\frac{7}{8}$. Mimi is convinced that $\frac{7}{8}$ is greater than $\frac{3}{4}$. Who is correct, Ronnie or Mimi? Support your answer using words, math and/or diagrams.

B. Josh is convinced that $\frac{3}{4}$ is greater than $\frac{19}{24}$. Jill is convinced that $\frac{19}{24}$ is greater than $\frac{3}{4}$. Who is correct, Josh or Jill? Support your answer using words, math and/or diagrams.

C. Using your responses from Part A and Part B, order the fractions $\frac{3}{4}$, $\frac{7}{8}$,and $\frac{19}{24}$ from least to greatest. Support your answer using words, math and/or diagrams.

Figure 7. The first page of the MDTP Written Response Item *Fraction Order*.

The prompts for written response item in Figure 7 are structured so that students must determine order between two values in Parts A and B. Students in the focal class experienced high success ordering three values, with 100% earning the full 4-points. Students used at least one of three strategies to complete the assignment: find common denominators, convert to decimals, and draw diagrams.

During the mid-course test (discussed in the next section), 73% correctly answered Question 5 (growth of 19%) with 15% selecting the incorrect option A (a decrease of 16%). Thus, this WRI served as a successful and timesaving re-engagement learning experience.

5.5. Mid-course Growth. Using diagnostic results to determine strengths, unfinished learning, and potential content gaps, and then enacting strategies to optimize instructional time, integrate topics, and distribute the learning over time were applied throughout the course. By the end of week four, the students had completed the first two units and two lessons in Unit 3 (Exponential Representations and Functions). To assess mid-course growth on MDTP topics, the AR test was administered during week five. Figure 8 displays the average percent correct on topics of the pre-course AR (blue) and the mid-course AR (black).

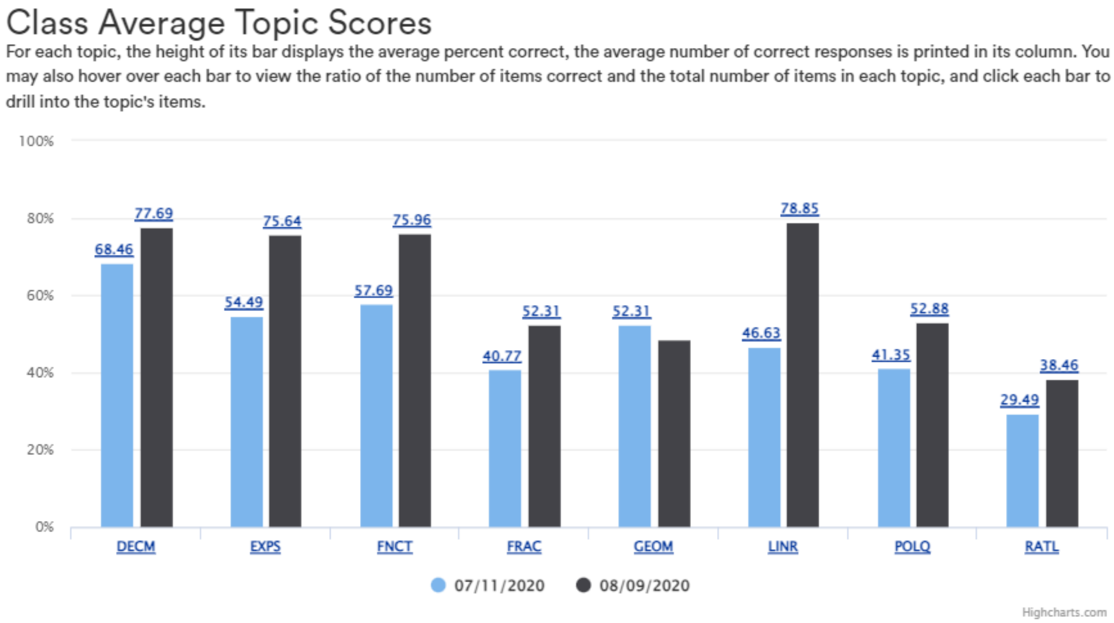

Figure 8. Topic Comparison from Pre-test to Mid-course Administration of the AR (as seen on the MDTP online platform).

Students demonstrated improvement in each MDTP topic except GEOM, with the most growth in the topics LINR, EXPS, and FNCT. Additionally, prior to Unit 2 (Linear Representations and Functions), only 12% of the students had met the Critical Level (CL) in the LINR topic (see Figure 2). In mid-course, 73% met the CL, a growth of 61%. These results indicate that more students were better prepared in most pre-course topics at mid-course.

6. Conclusion

The CSU/UC Mathematics Diagnostic Testing Project (MDTP) diagnostic tests and online testing platforms are tools to help teachers quickly identify students' strengths, unfinished learning, and potential gaps of content knowledge. The emphasis in this article has been on using the data from appropriate MDTP assessments of preparedness or course-level readiness tests in accordance with the MDTP Formative Assessment Framework (FAF), which requires **Identifying** students' current mathematical knowledge, **Analyzing** the mathematics students need for mastery of the content, and **Enacting** strategies to support learning goals. A particular focus of this discussion has been the use of MDTP diagnostic results within the FAF for a post-secondary, entry-level mathematics bridging course to precalculus.

As described in the background and overview seciton, MDTP's charge has been guided by mandates from the state education code. Thus, the Trustees of the California State University administer this diagnostic project in cooperation with the University of California, the California Community Colleges, and the State Department of Education. Their duties are to oversee the **development** of materials and services designed to assess pupils' knowledge and skills in the area of mathematics consistent with the Mathematics Framework for California Public Schools and the expectations of postsecondary education; the **distribution** of these materials and services to mathematics faculty members of the public school system upon request; and **support** for mathematics faculty of schools operated by school districts throughout California, especially at secondary schools with low student participation or achievement in postsecondary education.

All MDTP materials are thoroughly field tested to ensure reliability and validity after careful statistical analysis. Consultants from the Educational Testing Service (ETS) assist the Workgroup with review and maintaining face validity. MDTP tests are revised regularly to accommodate changing content standards.

MDTP provides free assessments, resources to assist best practice, pre- and post-data workshops, and regional conferences to assist mathematics educators to use MDTP in effective and appropriate ways. To access MDTP assessments, order paper test versions, and receive detailed information about MDTP, visit the MDTP website at https://mdtp.ucsd.edu. To schedule an MDTP data workshop or request to have an MDTP Director visit your school site, contact your MDTP regional office to receive personalized training and support.

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Research to Practice Sampler

Assessments

Kimberly Cervello Rogers and Sean P. Yee

Abstract. Research to practice samplers provide a short review of the research literature on a topic and then offer some examples of professional learning activities that leverage the research. In this case, the topic is types of assessment common in secondary and post-secondary mathematics, with a focus on preparing novice college mathematics instructors (e.g., graduate students and others first learning to teach in colleges and universities).

Assessment is a topic that includes statewide tests, end-of-semester exams, smaller-scale classroom tests or quizzes, and in-the-moment inquiries into students' learning. Assessments can provide instructors and students with information about students' mastery of concepts and where misconceptions or knowledge gaps may occur. It is vital for both novice and experienced instructors to utilize a wide variety of assessments to paint a clear picture of students' understanding and guide instructional decisions.

> Formative assessments are primarily used to inform the direction in which instructors might modify their lessons, summative assessments are conducted with the purpose of evaluating student proficiency with regard to one or more learning outcomes. (*MAA Instructional Practices Guide*, p. 53)

As in the quote above, two commonly known types of educational assessments are formative and summative. On the one hand, summative assessments are given periodically to determine at a particular point in time what students know and do not know (e.g., end-of-unit tests, end-of-semester exams). Summative assessments are assessments OF learning used as a means to gauge, at a particular point in time, student learning relative to content standards. Formative assessments, on the other hand, are part of the instructional process and provide information at the classroom level that can help instructors make adjustments and interventions during the learning process. Therefore, formative assessments are considered assessments FOR learning (Gold et al., 1999).

Consider the road test required to receive a driver's license. Every instance of practicing driving with a responsible adult or drivers' education instructor is important for learning, but not typically given a grade. However, if the road test outcome was an average of all the "grades" you received while practicing driving, low grades received when first practicing driving would affect your final grade. This driving test average would not accurately reflect your ability to drive, not provide constructive feedback, and not instill confidence in the assessment process. Instead, the

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final road test (summative) should serve as a measure of whether or not you have the skills necessary for a driver's license, not a reflection of all the (formative) practice that led to it.

Distinctions between these types of assessments are also apparent when considering the timing of assessments and how students may be involved in assessment processes. Formative assessments often focus on skills and concepts students are in the process of learning. Involving students in formative assessment processes can also increase students' motivation to learn (Fluckiger, et al., 2010). One component of engaging students in the assessment of their learning is providing them with descriptive feedback (Shute, 2008). Providing opportunities for novice instructors to practice and critique ways of efficiently and effectively providing descriptive feedback could help their students understand what they are doing well and how to improve their learning objectives. Formative assessments allow students to practice and learn from descriptive feedback, helping instructors and students collaboratively decide next steps in the learning process.

> Assessment can be roughly divided into two types, formative and summative, but these need not occur in isolation. In fact, many standard assessment techniques have elements of both. (MAA *Instructional Practices Guide* (IPG), p. 53)

In the process of learning two opposing concepts, learners often contrast the opposing concepts by generating a dichotomy or polarization (Henderson & Kesson, 2004). Many assessments have elements of both types; for instance, quizzes are often summative and formative because quizzes inform instructors whether they need to modify their instruction or reteach certain concepts while simultaneously evaluating students' proficiency with the content. To make the terms formative and summative meaningful, instructors need to progress beyond the either/or categorization of an assessment as formative or summative, to whether an assessment is more/less summative and more/less formative. This provides a more dynamic view of formative and summative assessments where the instructor deconstructs the dichotomy. In particular, seeing where an assessment lies on a formative-summative continuum can be used as a learning activity for novice instructors such as mathematics graduate students.

Activities for Professional Learning About Teaching

Note that these are activities for which *the learners are people who teach mathematics.* The tasks might be used in a workshop or seminar for novice college mathematics instructors (e.g., graduate students learning to teach undergraduate mathematics). Each activity is based on resources that are publicly available.

Activity 1: Discussing Vignettes of Assessments Being Used from MAA's IPG. Vignettes and commentary and discussion questions are included on pages 55–70 of the MAA's Instructional Practices Guide (IPG) focusing on different aspects of assessments for and of learning. Online resource (freely available) is the *MAA Instructional Practices Guide*.

Activity 1 Goals: Participants will

- Engage in discussions about different features of summative and formative assessments highlighted by vignettes.
- Design summative and formative assessments for specific learning objectives.
- Describe how each instructor in the vignettes adjusted their lessons to adapt to students' current understanding (see for example the summary discussion on p. 56 of the MAA's IPG).

These goals directly relate to the ideas described on the first page of this document because participants have the opportunity to get a better understanding of features and ways to use both types of assessments (formative and summative). Applying what they learn from the activity, participants could then more accurately use a variety of assessments types in their teaching to get a clear picture of students' understanding and guide their classroom decisions.

The MAA IPG provide vignettes related to, respectively, three sub-topics about formative assessments and three sub-topics about summative assessments (p. 55–70). Facilitators of teaching-focused professional development could select the vignettes and discussion topics that are most relevant for their novice-instructor-learners (e.g., TAs). If needed for timing purposes, participants could be asked to read the vignettes on their own before coming together during a seminar/class time to discuss their responses to the discussion sections and questions related to the selected vignettes. Seminar and class time could also be devoted to participants designing and modifying their assessments based on their reactions to the vignettes and ensuing discussions.

Activity 2: Dynamic View of Summative and Formative Assessments. The lesson plan and handout for this activity was designed for use with novice college instructors (Yee, 2017). It is accessible at http://seanpyee.wixsite.com/professional/resources.

Activity 2 Goals: Participants will

- Explain the differences and similarities of conceptual/procedural understanding as well as formative/summative assessments.
- Generate a formative/summative assessment while clarifying conceptual/procedural knowledge.
- Deconstruct summative/formative and conceptual/procedural dichotomies. Justify their position of their assessment using words such as "more summative" or "less formative."

The purpose and goals of Activity 2 build on the ideas described on the first page of this document in the following ways: The overall purpose is to have participants communicate and use the language of conceptual and procedural knowledge with formative and summative assessments. Then, they should be able to communicate to others why and how they perceive assessment and knowledge. This builds on the MAA IPG's explanation of the definitions of these types of assessments by having participants apply their understanding of the definitions in the context of classes they are teaching. Moreover, it seeks to help them unpack the apparent dichotomy behind the formative and summative assessment definitions.

There are four main components to the lesson. Two are preparatory. First, before seminar/class, participants develop some understanding about distinctions between the pairs of terms summative and formative, conceptual and procedural (see the lesson plan and refer to MAA's IPG pp. 49-64). Second, there is a discussion about these terms, and the reading, before the launch of Activity 2. Third, initiate Activity 2 with participants in groups of two or three and have people put their work on the board. Fourth, bring everyone together and have a discussion, during which participants provide justifications of their work.

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Book Review

Choosing to See

Review of: Pamela Seda and Kyndall Brown (2021). *Choosing to See: A Framework for Equity in the Math Classroom.* Dave Burgess Consulting.

For more than a century, the goals for schooling in the United States have been a balancing act between efforts to reflect and efforts to shape society (Dewey, 1907). Throughout its post-industrial history, most American schooling—from teacher preparation and curricula to building designs and management structures—has relied on tenets that see "the students" as a collective mechanism for maintaining existing social norms. Choosing not to see the many different humans who make up the students in a room has meant many are outsiders in their own classrooms. *Choosing to See: A Framework for Equity in the Math Classroom*, by Pamela Seda and Kyndall Brown, is for teachers and others who seek ways to work with those humans in their classrooms. The title refers to acknowledging the situation and the subtitle is about how to deal with it. The foundation of the book is both a philosophy and a practical framework for going about it.

The philosophy is that mathematics is culturally in Drive (not Neutral) and that our job as teachers includes seeing how mathematics curricula can be updated to reflect the backgrounds and experiences (in context) of students. What does that look like in a mathematics classroom? How do you start and sustain it? The book's ICUCARE framework is illustrated with many examples (and non-examples) throughout the chapters:

- **I** include others as experts
- **C** be critically conscious
- **U** understand your students well
- **C** culturally relevant curricula,
- **A** assess, activate, and build on prior knowledge
- **R** release control
- **E** expect more.

The suggestions go from easy to use baby steps for increasing student-relevance, like putting our students' names, school names, other teachers' names, into problem contexts, to more extensive work that includes interviewing students, having listening conversations, to engaging with local communities. The book includes ideas for how to be a teacher who decides to do each of these AND how to implement a student version of the framework within the classroom itself. For example, *I: include others as experts* has a student version: look beyond the expertise of the teacher to recognize your own competence and that of your classmates.

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The writers bring their own multi-faceted background experiences to this book. They urge teachers to do candid reflection about their ways of seeing (or not) the various aspects of humans (e.g., race, gender, ability, spiritual and cultural value systems). The book calls on readers to recognize the values and attitudes prevalent in schools, and how they affect students. For example, internal dialog is encouraged through self-questioning: "How am I reacting to students in ways that confirm my biases? How am I purposefully being aware and challenging my familiar routines?"

Students who share a teacher's biases often suffer because of them. If you believe that your group is no good in math, you are primed with low expectations for yourself and a built-in excuse for doing poorly. If you are from a group that is supposed to be good at math but you find the subject difficult, you easily see yourself as a failure. Teachers must do more than recognize insidious cultural biases in the classroom. They must actively combat those biases. The authors present examples of how to do so, from simple shifts in practice like citing black or Latino mathematicians and their contributions to more substantive reshaping of professional behaviors.

Confronting unproductive assumptions and related biases is just a first step. Whether because of cultural biases or other reasons, students who feel that they are no good at math often avoid engaging in the material. The authors offer many ideas to disrupt and revise this feeling, including strategies to support learners in their own seeing, in particular a sense of their own value as a mathematics student. The authors describe these approaches in detail, using specific examples, many from their own classroom experiences (e.g., the story of Jasmine).

The authors challenge teachers who read the book to carefully examine and adapt their classroom habits. The authors argue that the key steps for teachers are to recognize and to actively accept the challenges inherent in thinking about the work we do and how it is rooted in our own expectations as well as those of students and the assumptions behind school structures and policy. The premise of the book is that meeting this challenge will go a long way to solving the problem of inequity in mathematics classes.

Choosing to See is well written and carefully documented. I was so engrossed in it that I missed my subway stop!

Still, I am left with a few questions that I wish had been explored in more depth in this book:

(1) Most of the examples involve African American and Latino students. In California those are two of the largest groups who are of concern. I would have liked more attention paid to choosing to see other groups such as those of Asian, Hawaiian/Pacific Island, or Native American backgrounds. For example, Native American students face similar issues to those discussed in the book, and their alienation from public schools is at least as great as that of African Americans and Latinos. However, their school circumstances, about 24% of this demographic group, are in poor, remote areas, quite different from the urban and suburban settings most referenced in the book.

(2) I wonder about the emphasis on verbal explanation. I have seen so many students whose linguistic skill levels are different from their mathematical skill levels. I have been convinced that students know what they are doing, even when they are at a loss to explain. A first grader I saw, asked to add 55.6 to itself, immediately said 111.2. It would be hard to get that number without some real understanding, but when an adult

asked him to explain it, his English language skills allowed for a two-word response: "I added." Insisting on a more detailed explanation only frustrated him. Still being explored by research and development are questions about how success in mathematics classes is intertwined with, possibly dependent on, success in verbal expression. The fact that there are so many unknowns is particularly hard on students whose native language is not English.

(3) I wish the book (or a maybe a sequel to it) could deal with differences of another kind: speed and interest. Sometimes there are students who move much faster than others. Asking the speedy kids to work in groups with classmates or to help teach them is a very short-term solution. Without something more to keep them engaged, they can get bored or even cause discipline problems.

One approach that is mentioned in the book is to use problems with "low thresholds and high ceilings," an approach exemplified in the recent work of Jo Boaler, Robert Berry, and many others (Bartell et al., 2023; Berry et al., 2020; Boaler, 2020; Conway et al., 2023; Koestler et al., 2022; Matthews et al., 2022). The aim is making sure everyone succeeds at the basic lesson (threshold) while also offering those who are ready a way to explore the high ceilings.

As an example, consider the following: What numbers can you make by adding two or more consecutive whole numbers? Once the term "consecutive" is clarified, learners can start to experiment. They soon find they can make 3, 5. 6, 7 and 9 but not 1, 2, 4, or 8. Finding what numbers up to 40 can be made that way is good practice with basic addition and may lead to observations and conjectures. (e.g., "you can make every odd number" or "the next number you can't make is 16"). Students can work in groups, checking and helping each other. But even with group work, some kids will want to move ahead much more quickly than others. To keep them involved, they can explore how many ways each number can be made. The answer is 0 for 1, 2, 4, and 8, but it is 2 for 9, since 9 can be made two ways as 2+3+4 and as 4+5. To find the pattern is a deeper question (Hint: How many odd numbers divide the number?). That more difficult question will hold the focus of those who are ready for it and free the teacher to work with those who need help.

In closing I will note that the slogan "Every child can learn" affirms the potential within each student even as it reflects the fact that many students don't learn very well in our schools. It is easy to blame external factors like skimpy budgets, crowded classrooms, students coming to school unprepared to learn, and so on. *Choosing to See* addresses critical aspects of equity—noticing and being intentional in responding to the noticed things. The book offers concrete ways to do both. This is a valuable book for teachers.

Contributed by: Bob Stein.

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