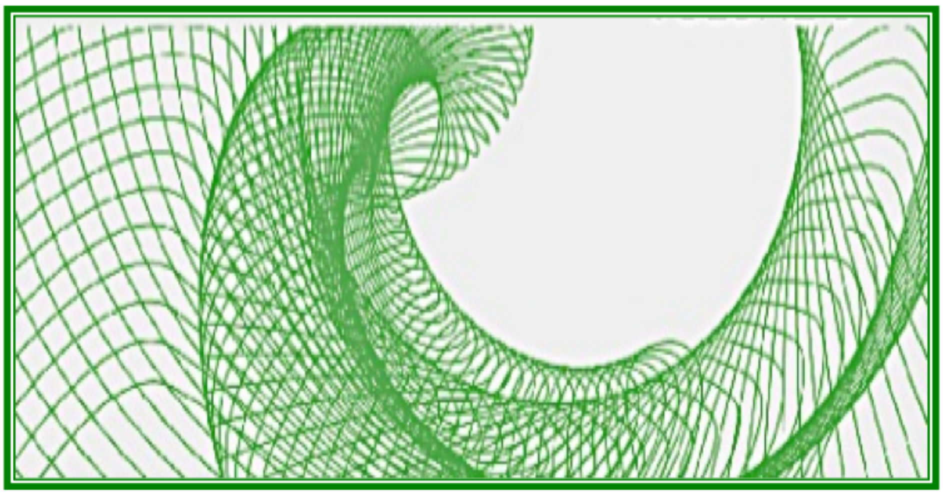
Journal of the California Mathematics Project



California State University Stanislaus

Volume 7 Summer 2015

Introduction

The Journal of the California Mathematics Project (JCMP) is a supplementary publication of the California Mathematics Project (CMP) and is sponsored by CSU Stanislaus. The official news publication of the CMP is the *California Online Mathematics Education Times (COMET)*, an electronic newsletter edited by Professor Carol Fry Bohlin at CSU Fresno.

The JCMP editorial mission is communication about research and practice among educators who are engaged in the California Mathematics Project or similar initiatives, in California and elsewhere. Authors include K-12 teachers who are particularly interested in mathematics education, higher education faculty, and others doing research and development around mathematics teaching, learning, and policy. The journal also publishes articles by higher education students as a way to encourage research and foster professional development. Articles in the journal cover mathematics across the entire K-16 spectrum and teacher preparation and development for the same. Each volume is a mix of articles.

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Editorial Board

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Volume **7**, 2015

A Pedagogical Framework for Distinguishing Mathematical Definitions in the Classroom

RaKissa Cribari

Abstract. Most students believe that definitions play an important role in the learning of mathematics. However, they struggle with transitioning from performing procedures to reasoning from definitions as they move into more sophisticated mathematics. Further, students tend to focus in on their experiences, examples, and intuition, especially the visual/mental images they hold for a concept, instead of the axiomatic assertions provided by a mathematical definition. This disconnect between approaches needs to be addressed at all levels of study. The goal of this paper is to present a pedagogical framework and some examples on how to address the role of definition in the study of mathematics for students at all levels.

Introduction

What is the definition of a circle? My class of pre-service teachers agreed that a circle is the set of all points equidistant from some center point C. Through some class discussion they realized it was important to clarify that this definition is inaccurate if we do not state that it holds on a two-dimensional plane. Immediately after the conversation about the definition of a circle ended I asked the class to give me examples of circles. Cup! Soccer Ball! Polka Dot! Stop Light! Coin! They shouted these out with glee as I noticed they were completely unaware of how their mental image of circle overrode the key points of a definition we just spent five minutes deliberating. How could I help my students recognize that being sloppy with mathematical definitions is not going to bode well for them as teachers or learners of mathematics?

While attending primary school I was taught a variety of definitions in mathematics. However, most of the definitions that were used were more informal in nature (e.g., a triangle is a three sided figure). It was not until I was an undergraduate majoring in mathematics taking a linear algebra class that I discovered the importance of more formal definitions in mathematics.

This discovery did not come from my professor stating explicitly to my class the necessity of learning mathematical definitions or their importance in the learning of linear algebra. Rather it was born out of the fact that I was getting a C in the class and really had no idea why I was performing so poorly. I sought out the help of my professor and together we discovered that I completely ignored the definitions at my disposal. In fact, I was not even clear on how they would help me to do my homework. It was at that point that my professor clarified to me not only the importance of knowing what the definition states but what information it tells me about the concept itself. It was made clear that understanding the formal mathematical definitions would be critical in not only helping me to solve my homework problems in his class but were also necessary for me to be successful in all of the upper division mathematics courses I would be taking due to the proof writing aspect of the courses.

This was a huge *Aha!* moment for me in my development as a learner of mathematics. And I believe that it has been a bit of a rite of passage for many scholars in mathematics. However, I do not believe that professors of mathematics knowingly keep this information from their students. In fact, the use and importance of mathematical definitions is so deeply engrained in their culture that it is likely assumed their students already know the role and use of mathematical definitions. This is true even though professors of mathematics also recognize that their students do not "know" mathematical definitions (Alcock & Simpson, 2002; Edward & Ward, 2008). I would also posit that because the role and use of mathematical definitions is not made clear to students, it could be a reason why any first course in mathematical proof (for me it was linear algebra) tends to be the course that weeds out mathematics majors from completing a degree in mathematics.

Background

The creation and use of mathematical definitions is quite different from our "everyday language" definitions. Edward and Ward (2004) distinguished two different types of definitions: extracted (everyday language) and stipulated (mathematical) definitions. *Extracted definitions* are inductive, "based on examples of actual usage, definitions extracted from a body of evidence" (Landau, 2001, p. 165). Whereas *stipulated definitions* are axiomatic assertions, "setting up of the meaning-relation between some word and some object, the act of assigning an object to a name (or a name to an object)" (Robinson, 1962, p. 59).

In their study, Edward and Ward (2004) found that undergraduate mathematics majors often do not view a mathematical definition as stipulated like a mathematician would. Moreover, it was discovered that students would choose an extracted definition approach, and their intuition about a concept, over a given, stipulated definition, when reasoning about mathematical tasks. That is, the participants in the study stated and explained the stipulated definitions needed to perform a mathematical task, but were unsuccessful in their attempts to complete the task. Knowing how to state definitions is not enough to be successful in performing mathematical tasks, especially those involving proof writing.

Pre-service Elementary Teachers

My experience training prospective elementary school teachers has revealed similar ways of thinking about definitions in mathematics. While pre-service teachers believe that definitions play an integral role in the learning and teaching of mathematics, they notoriously ignore the constraints in a stipulated definition. When reasoning, they rely primarily on their experiences, examples, and intuition, especially the visual/mental image they hold for the concept. For example, while teaching a Geometry and Measurement course for pre-service elementary teachers I am regularly surprised by scenarios like the one at the start of this paper. None of the examples of circles. Rather, they are examples of items that have a circular quality. This exemplifies how the extracted approach will often take precedence over the stipulated, rigorous mathematical definition when working on mathematical tasks.

In a study among pre-service teachers, Ward (2004) posited that the future teachers in her study needed a mathematical "intervention." Specifically Ward recommended that students should see examples of concepts that are not always traditional in appearance. She suggest that unconventional examples would help to develop an approach to noticing and using definition that is more consistent with standard mathematical use. Further, Ward suggested that teacher educators teaching mathematics content courses are on the front lines, facilitating the breaking of the cycle. I would go one step further and suggest that the disconnect between approaches to definitions needs to be addressed at all levels of study in elementary, secondary, and post-secondary mathematics. The goal of this paper is to present a framework, with some examples about key aspects of definitions, and offer an activity to address the role of definition in the study of mathematics for students at all levels.

Pedagogical Framework

The role of definition must be explicitly discussed with students. Caution is needed in determining what students understand about the role of definition in mathematics. Many students believe that without at least a basic understanding of a definition it can be hard to solve problems involving those definitions. However, few have experience with understanding and working from mathematical definitions as a major aspect to being successful in mathematics. Therefore, educators must create opportunities for students to see the importance of mathematical definition not only in communicating mathematical ideas but, more importantly, in reasoning from those definitions to construct conjectures, provide sound mathematical arguments, make connections between concepts, and to discover new mathematics (Cuoco, Goldenberg, & Mark, 1996; Leikin & Zazkis, 2010).

Repeated exposure to the importance of mathematical definition is necessary. I was once told that a belief is just a thought that you thought a lot. Educators can play a central role in helping to shape student beliefs about the role of definition in mathematics by implementing frequent opportunities for students to explore mathematical definitions. In fact, this can be a thread woven throughout a course. The sooner and more times students are exposed to mathematical definitions as a mathematical habit of mind, the better.

Distinguishing a mathematical definition is an essential mathematical habit of mind. Exploring mathematical definitions can have many facets. One method that I focus on here, and that I have found the most powerful in my classrooms, is to distinguish the definition. Distinguishing a definition is quite simply determining what a mathematical concept is and what it is not, identifying examples and non-examples of it. Distinguishing a definition provides an opportunity to tease apart the necessary and sufficient conditions of a mathematical concept so students will have an understanding of the edges and constraints of a defined concept.

What is more, distinguishing a definition can assist the learner in turning abstract ideas into working knowledge. This is accomplished by focusing student attention on each of the necessary and sufficient components of a definition and exploring how augmenting or removing any one of the conditions alters the concept in a nontrivial way (Cuoco et al., 1996). This process plays a significant role when determining when objects are members of a category. This is particularly valuable when reasoning from definitions in a proof writing context.

Providing non-traditional examples and non-examples of concepts helps students to gain a richer and more nuanced view of the concept. Non-traditional examples include extreme cases of the concept or non-traditional orientations of the concept (e.g., non-gravity based and/or irregular shaped polygons).

For example, when distinguishing box and whisker plots with students instead of providing only a traditional example (see Figure 1), educators can also include several non-traditional examples that will foster a broader view of a box and whisker plot (see Figure 2, next page).

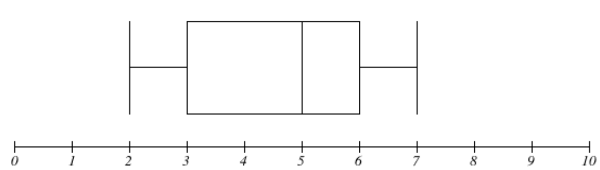

FIGURE 1. Traditional example of a box and whisker plot.

In Figure 1 we can see that the five number summary has all distinct values. However, in each of the examples in Figure 2 we can see that something peculiar is happening. Most students will not connect that oddity to the five number summary. Students' natural assumption is that the plots in Figure 2 are non-examples. With some questioning about what it might mean about the data if there is a missing whisker, students will eventually come to the conclusion that the third quartile and the maximum in the lefthand plot in Figure 2 must share the same value and therefore the upper 25% of values might all be the same value, namely 5. In the righthand plot in Figure 2, students begin to realize that not only are the

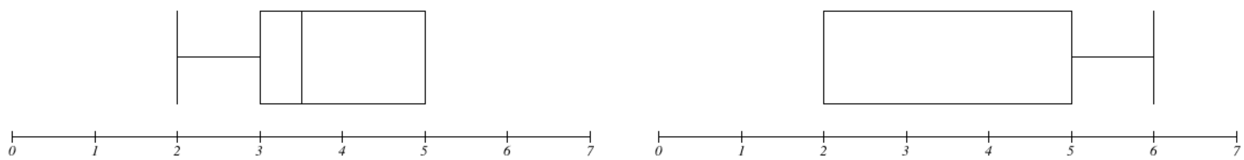

FIGURE 2. Non-traditional examples of box and whisker plots.

first quartile and the minimum equal but the median must also share that same value. Further, the lower half of the data must consist of all the same values, in this case 2.

Be intentional with definitions in the classroom. Illustrating the misuse of a definition can provide a compelling argument for being intentional with our word choices in a mathematics course. In the classroom, this might look like choosing our words carefully and intentionally to create cognitive dissonance for our students. For example, when I introduce measures of central tendency I purposely provide my students with a classic example of using the word "average" to confuse and mislead. Most people take average to be the same as the arithmetic mean. However, through a whole class discussion students come to understand that the word average can refer to mean, median, or mode. This conversation provides an opportunity for students to see how important it is to say what they mean and mean what they say in a mathematics classroom. Intentionally providing opportunities for students to see a misuse of a definition can help them to come to understand rather quickly that they need to be precise with their word choices in a mathematics classroom when referring to mathematical concepts.

This four pronged pedagogical framework of attending to (a) the role of definition, (b) repeated exposure to the use of definitions as problem-solving and reasoning tool, (c) distinguishing a definition, and (d) intentional use of language in working with definitions in challenging ways, requires educators to make mathematical definition and its importance to the learning and understanding of mathematics a pervasive theme in the classroom. Further, the framework promotes active and deep engagement with definitions to encourage reasoning and sense making from definitions.

Coochy-Coo Activity

To provide a meaningful experience regarding the role of definition in mathematics for a workshop with in-service elementary teachers, I developed the Coochy-Coo activity. The activity places students in the unique position of learning mathematics using a new language in which they must be able to communicate and reason. This activity brings students through three different levels of understanding of a mathematical definition. The first level is the informal definition, the way a student might think of the concept focusing on the most basic aspects of the definition. The second level is the typical school or textbook definition. This level is slightly more rigorous than the informal definition. However, it may lack some of the necessary and sufficient conditions that the third level of definition would entail, the formal mathematical definition, that is used to reason about mathematical concepts.

In the Coochy-Coo activity, students are working to understand what a coochy-coo (i.e., a heptagon) is in a new language. They are given a student definition that purposely does not use the word polygon and are asked to determine from a group of figures which ones are coochy-coos based on this informal definition. Next students are given a textbook definition that is filled with even more words with which students are not familiar. Students are then provided with additional textbook definitions in English to help clarify what they meanings of terms are. For example, students are told that "A coochy-coo is a tiddly-wink enclosed by seven biggity-bee chack-taks." Students are then provided with definitions of tiddly-wink (a region) , biggity-bee (straight), and chack-taks (line segments). They must piece these together to create a working definition of a coochy-coo so they can further distinguish it from a group of figures using this new definition.

Lastly, students are provided a formal mathematical definition completely in English, though it may contain words that are new to (e.g., collinear or closed polygonal path). The formal definition has all the necessary and sufficient conditions to identify a figure as being a coochy-coo. Students then reexamine all the figures to identify which are coochy-coos using the formal mathematical definition. The purpose of requiring students to go through all three levels of definition is to assist them in recognizing that mathematical definitions build on each other and one must understand all of the mathematical terms in a definition to truly understand the new concepts being defined.

The last task for students is to come up with their own examples and non-examples of coochy-coos. The purpose of this part of the activity is to further facilitate a student's ability to distinguish definitions through developing unique nontrivial examples and non-examples. The activity further encourages students to start looking at non-traditional examples of definitions. This fosters an awareness that will be more consistent with the formal mathematical definition.

The Coochy-Coo Activity has been successfully implemented with in-service and pre-service elementary teachers. Both groups struggled with the formal definition of a coochy-coo (heptagon). Their struggles centered on whether shape E would be considered a coochy-coo or not (see Figure 3).

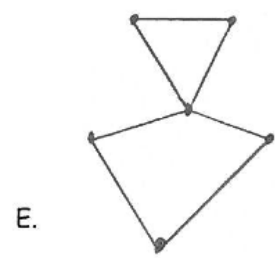

Figure 3. Shape E: Is this a coochy-coo?

Their intuition said no, but it was challenging to figure out how to use the definition to eliminate it. Specifically, they were not reasoning from the definition of a closed polygonal path which states that all points (vertices) must be different (i.e., unique). The vertex that joins the triangular figure to the quadrilateral figure is the issue in this particular shape. Notice that there are four line segments that utilize that one vertex thus the shape is not a closed polygonal path and, therefore, it follows that the figure is not a coochy-coo.

Another troublesome area for some students was coming up with unique non-examples of coochy-coos. Students would initially create figures that were closed or did not have seven sides – making their examples trivial and less interesting. This provided a great opportunity to discuss why it would be important to develop examples that have all the necessary and sufficient conditions of the definition but one subtle piece is missing to help fully distinguish the definition. In our class discussion, I further connected this idea to how new mathematics can be created by examining something familiar but removing a condition that forces it to operate in a new way.

I have noticed my students' relationship to and perception of mathematical definition has changed dramatically. Certainly, it is at the forefront of their minds in my courses. For example, in my geometry course for pre-service teachers, I now find more students refer to specific components of a mathematical definition to support their reasoning about a problem situation. In past semesters, students solely relied on their loosely connected experiences and intuition in an extracted definition approach to reasoning through similar scenarios. In fact, many students have commented on how surprised they were that none of their other mathematics teachers had pointed out how important definitions were in mathematics. The majority of my students stated on midterm evaluations that a focus on definition as a constant component of the work made them more aware of how powerful the definitions were in helping them to be successful in the course.

Implementing the Coochy-Coo Activity early in the semester (or early in a workshop) has helped me to set the level of mathematical rigor required for success in the course while also highlight the importance of definition in the learning and understanding of the material. Mathematical definition has become a theme woven into my courses, strengthening my students understanding of concepts.

Implications for Practice

Employing this pedagogical framework will require teachers to re-examine the curriculum to identify prime locations for implementing activities throughout the content that bring the stipulated mathematical definition to the forefront of learning. Additionally, it means that educators will need to practice refraining from providing the most typical examples of a concept as the first examples. Mathematics educators will be called upon to generate examples in the classroom that highlight the nuances of the mathematical concepts they are responsible for teaching.

The idea of providing more interesting and nuanced examples in the classroom may be challenging for many pre-service and in-service teachers. Therefore, teacher educators will also need to bring attention to stipulated mathematical definitions. Further, they will need to provide opportunities for educators to examine these definitions and to practice developing and identifying thought-provoking examples and non-examples.

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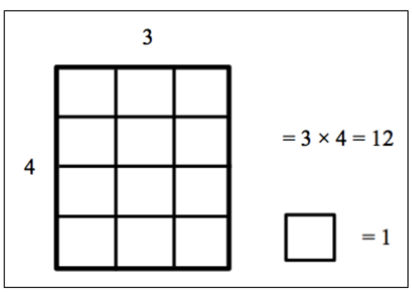
Journal of the California Mathematics Project
Volume **7**, 2015

Deepening Student Understanding of Area and Volume by Focusing on Units and Arrays

Allison Dorko and Natasha Speer

Introduction

When we teach elementary school students about area, surface area, and volume, we are teaching ideas that help them measure the physical world. In biology and chemistry, the surface area to volume ratio sets a limit to the size of cells, and is also a factor in the rate at which chemical reactions occur. Volume is important in measuring density, and in calculus, understanding area and volume is critical for understanding the construction of Riemann sums that are used to model accumulation of various quantities. In addition to these applied uses in science, technology, engineering and mathematics (STEM) fields, by teaching students these ideas, we are also giving them tools with which to understand other mathematics. For example, beyond serving as a measure of space, area can be used as a model for multiplication of real numbers and multiplication of expressions with variables (see Figure 1).



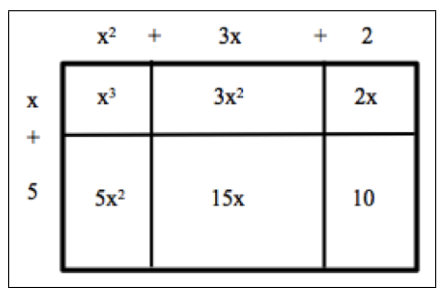


Figure 1. Examples of area models as they are sometimes used to illustrate multiplication of numbers (left) or multiplication of algebraic expressions (right).

The Common Core State Standards recommend that mathematics in early childhood focus on number, geometry, spatial relations, and measurement (Council of Chief State School Officers [CCSSO], 2010). Included in this is understanding units of measure, which are notoriously tricky for students of all ages. Given that units of measurement can serve as clues about how to interpret and combine quantities, helping students understand and make use of units gives them a tool for problem-solving and mathematical modeling.

Students of all ages can have difficulty understanding these important ideas and even those who carry out computations accurately may not have a rich understanding of the concepts of area and volume (Battista & Clements, 1998; Simon & Blume, 1994). We know that many students do develop good understanding of these ideas by the time they reach college. However, those who do not may face particular challenges when the instruction they receive makes substantial use of units, area, and volume. To better understand these challenges, we have been investigating college students' understanding of area and volume and associated units of measurement. This work provides mathematics instructors with information about student understanding and can be used to identify the difficulties that persist for some students. In particular, we have used it to inform the design of activities for secondary school students that may help strengthen student understanding of area, volume, and their measurement.

Research Findings

Area and volume computations are based on the idea of iterating unit squares (or cubes, as appropriate[1]) into rows and columns (or rows, columns, and layers) such that there are no gaps or overlaps. While adults may perceive these rows, columns, and layers as an organized structure, some learners may see an array of unit-sized pieces as randomly arranged objects (Battista & Clements, 1998). For these students, the number of unit-sized pieces do not represent a spatial measure. Difficulties with perceiving the structure of an array and with spatial visualization may lead to students not counting the innermost cubes of an array (e.g., counting 26 cubes in a 3×3 array like a Rubik's cube because the centermost cube is not visible) and double-counting the edge or corner cubes. In other words, students may see a three-dimensional array in terms of its faces and neglect the "middle" (Battista & Clements, 1998). Not perceiving area and volume as arrays may explain why many students, including preservice teachers, cannot explain why a formula like $A = lw$ generates a measure of area or why $V = lwh$ gives a measure of volume (Simon & Blume, 1994). Relatedly, identifying the shape of the cross-section of a solid can be difficult for middle school and high school students (Davis, 1973), which prevents them from thinking about the volume of a rectangular solid as the area of the base times the height. The good news is that students who do understand arrays tend to be successful with area and volume computations (Curry & Outhred, 2005; Dorko & Speer, 2013a, 2013b).

Units of measure are also difficult for students of all ages. Elementary school students tend to misappropriate units of length for area, volume, and angle (Lehrer, 2003). Researchers have replicated this finding with different tasks and concluded that some students do not see the need for a unit of cover in area measurements. Instead, students with this view will add lengths together to get the "area." For example, some children measure the area of a square by measuring one side of the square, moving the ruler a short distance parallel to the side of the square, and measuring the lengths of the two sides perpendicular to the ruler with each successive horizontal movement (Lehrer, 2003). Misappropriating length units for other spatial measures seems indicative of trouble with dimensionality (i.e., length is one-dimensional but area is two-dimensional). If a learner does not perceive area as composed of unit squares and volume as composed of unit cubes, then expressing areas and volumes in square and cubic units can seem arbitrary.

Though one would hope, or even assume, that undergraduates understand area, volume, and units, our research suggests otherwise. We began studying student learning about volumes of solids of revolution and optimization in calculus. It turned out that in many cases, students' sparse understanding of area, volume, and surface area prevented them from completing the mathematical modeling needed for using spatial accumulation ideas from calculus. To look more deeply into how students thought about area, volume, and units in non-calculus contexts, we gave basic area and volume computational tasks (areas of a rectangle and circle; volumes of a rectangular prism, a cylinder, and a right triangular prism) to 198 calculus students. Seventy-three percent of students gave an incorrect unit for at least one task (Dorko & Speer, 2014, in press). We found that students struggled with computing volumes. Some students computed surface area instead of volume, which is reminiscent of elementary school students' tendency to think about only the faces of an object when they enumerate a volume array. Finally, some students used formulae that contained both surface area and volume elements (which we term "amalgam" formulae). Table 1 (next page) categorizes some student responses to a problem that directed them to find the volume of a cylinder of radius r and height h. Note that some formulae fall into two rows. For example, some students said that $2\pi r^2$ found the area of the base, and multiplying by the height gave them volume (putting this in the second row) while others described the two as accounting for the "two bases" (a surface area idea).

[1]A "unit square" is a covering shape (interior and edges) in the plane that is bounded by four sides of equal length with four equal interior angles and a "unit cube" is the solid (interior, faces, and edges) bounded by six unit square faces with three meeting at each vertex.

Table 1. Student-Produced Formulae for the Volume of a Cylinder of Radius r and Height $h^\dagger$

Correct	**Incorrect**			
Volume Formula	*No surface area element*	*Amalgam of surface area and volume elements*	*Surface area*	*Other*
$\pi r^2 h$	$2\pi r^2 h$	$2\pi r^2 h$	$2\pi r^2 h{+}2\pi rh$	$d+h$
	$\frac{1}{3}\pi r^2 h$	$2\pi rh$	$2\pi r^2 h + \pi dh$	
	$\frac{1}{2}\pi r^2 h$	$2\pi r + \pi rh$		
	$\frac{4}{3}\pi r^2 h$	$\pi r^2 + 2\pi d$		
	πrh	$2\pi r^2 + 2rh$		
	$\frac{1}{2}\pi rh$			
	hdr			

† From Dorko and Speer, 2013b (p. 54).

Sometimes teachers and professors assume that students simply remembered a formula incorrectly. Our results indicate otherwise. The students we interviewed explained the formula they offered. For example, one student said, for $2\pi r + \pi rh$, that $2\pi r$ gave the measure of the areas of the bases and πrh accounted for the space between them. We suggest that teachers view an incorrect formula as a likely symptom of not understanding area, volume, and/or surface area. For instance, suppose a student used the formula $2\pi r^2 h$ to find the volume of a cylinder. In responding to the student's work, one might be inclined to cross out the two, assuming that the student "forgot" that the area of a circle is simply πr^2 and not $2\pi r^2$. If the student had written $2\pi r^2 h$ knowing that the area of a circle was πr^2 but thinking the volume calculation needed to "include both bases" (as some of our students explained for the $2\pi r^2 h$ formula), a teacher crossing out the two or writing the correct formula does nothing to help the student better understand volume. Instead, we suggest using students' formulas as clues to how they are thinking about volume.

In the next section, we provide other ideas for instruction. These ideas are based around connecting units of measure to other ideas in mathematics, such as rules of exponents, and on tasks that build students' understanding of arrays so that they can come to see the volumes of rectangular solids as the area of the base times the height.

Ideas for Instruction

Table 2 lists some of the findings from our research, the implications for instruction, and references to tasks that might be used to strengthen student understanding. The tasks are in the print-ready Worksheets that start on page 11. See page 14 for accompanying teacher notes.

Conclusion

In summary, we have seen that some of the issues that elementary, middle, and high school students experience regarding area, volume, and units persist into the undergraduate years. Of course, this is not true for all students. On the whole, undergraduates in our work were more successful with area and volume computations, and explaining area and volume formulae, than what research has reported for younger students. The undergraduates who were successful with these computations and who could unpack and explain a formula generally did so because they conceived of areas and volumes as arrays. We suspect that thinking about arrays makes it easier for students to understand why area is in units squared and volume is in units cubed. Early facility with units provides students with a tool to help them understand and carry out calculations in chemistry, physics, differential equations, and other areas of mathematics. In short, mathematical measurement is a way to see and appreciate the world.

Table 2. Research, Instruction, and Tasks for Classroom Use

Research Finding	*Instructional Implication*	*Tasks*
Students struggle with the units for area and volume computations.	Connecting squared/cubed units to laws of exponents provides students with an example of laws of exponents in the real world. For instance, show that the volume of a cube with side length 5 is ($5\text{ cm}^1 \times 5\text{ cm}^1 \times 5\text{cm}^1$) = 5^3cm^3. Emphasize that a linear unit has an exponent of 1, just like 7 can be written as 7^1 and x can be written as x^1, and hence the "when you multiply, you add exponents" rule applies to units as well as numbers and variables.	A, B
Students struggle with understanding arrays and that the volume of a rectangular solid can be thought of as $V = Bh$ where B is the area of the base.	Most volume computations that elementary and secondary school students are asked to complete can be carried out with an "area of base times height" approach. However, students often encounter specialized versions of this for shapes such as cubes, rectangular prisms, and cylinders (s^3, lwh, and $\pi r^2 h$ respectively). Emphasize that these formulae are special cases of the more general approaches that also better illustrate the idea of layers or arrays. For instance, writing the volume of a cube as $s^2 \cdot s$ may make it clearer to students that s^2 is the area of the base and s is the height.	C, D, E, F

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Worksheet Pages

A

Rewrite the following using exponents:

$5 \cdot 5 \cdot 5 \cdot 5 =$ ________

$a \cdot a =$ ________

$\text{cm} \cdot \text{cm} =$ ________

Expand the following as products so they are written using only terms whose exponents are 1:

$7^4 =$ ________

$x^7 =$ ________

$\text{ft}^3 =$ ________

Would it make sense for the area of this piece of paper to be 93.5 inches? Why or why not?

B

For each measurement given, pose a problem whose *answer* is the spatial measure listed. (You can think of this like the game *Jeopardy*, where you are given the answer and you need to design the question).

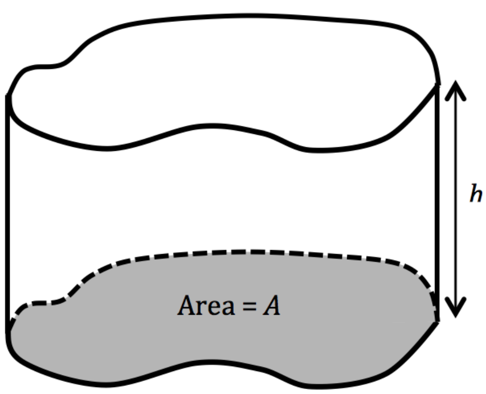
8 cm^3

4π cm

$9\pi \text{ cm}^2$

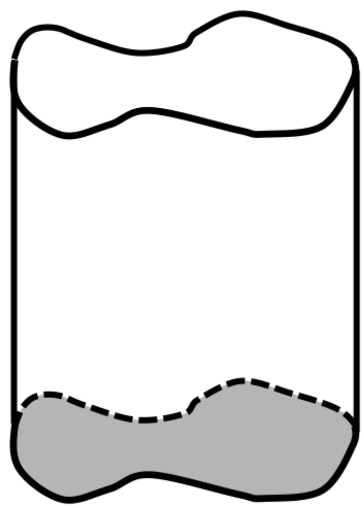
$9\pi \text{ cm}^3$

C



If the area of the base of this prism is 44.1 cm^2 and the height of the prism is 7.3 cm, what is the volume?

D



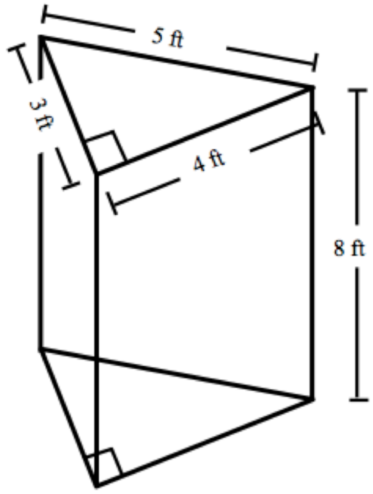
If the volume of this prism is 56 in^3 and its height is 7 in, then which of the following is the area of the base?

(a) 8 in (b) 8 in^2 (c) 392 in^2

(d) 392 in^3 (e) 8π (f) $8\pi \text{ in}^2$

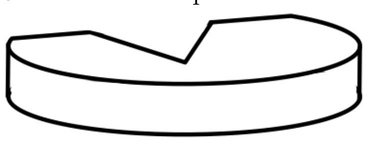
E

Find the volume of the triangular prism pictured below.

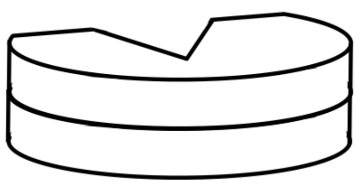


F

Consider the shape below. The area of its base is 6 in^2 and its height is 1 in. What is its volume?



Now suppose we have many of these shapes, and we stack them to form bigger shapes.



If we stack two of them together, what is the area of the base?

What is the height?

What is the volume?

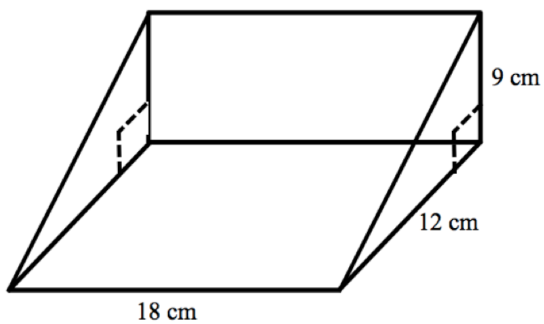
Suppose we continue to stack these shapes. Complete the table below.

Number of Layers	Area of Base	Height	Volume(Area of Base $\times$ Height)
1			
2			
3			
4			
$\vdots$			
n			

What is the volume of the shape formed by stacking n layers?

G

While the word “base” in English usually means the “bottom,” in mathematics the base of a shape may not actually be the face that looks like it is resting on the “ground.” For instance, the prism below is considered a *right triangular prism* with a base that is a triangle.

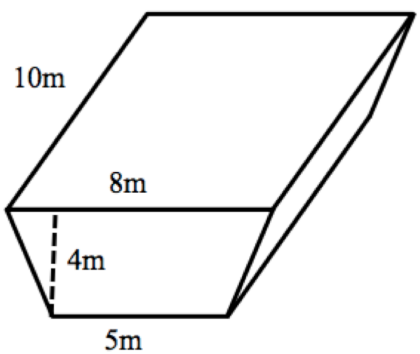


(a) What is the area of the base of this prism?

(b) What is the volume of this prism?

H

Consider this prism:



(a) Shade the prism’s base.

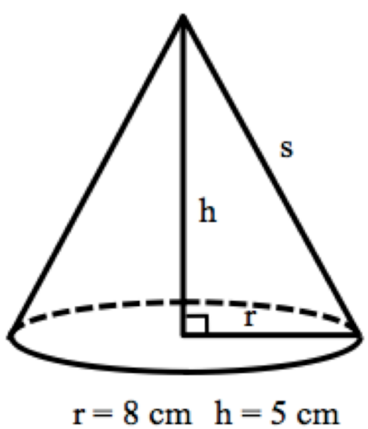
(b) What is the name of this prism?

(c) Find the area of the base.

(d) Find the volume of the prism.

I

A friend looks at the following shape and tells you that the area of the base is 25π cm^2 and that the volume is $25\pi \text{ cm}^2 \cdot 8 \text{ cm} = 200\pi \text{ cm}^3$.



(a) Did your friend find the correct volume? Why or why not?

(b) Does your friend’s computation have the correct units for volume? Why or why not?

Teacher Notes on Worksheet Tasks

<table>
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A

For the tasks:

cm · cm = _______
ft^3 = _______

some students may answer:

cm · cm = c^2m^2
$\text{ft}^3 = \text{f}^3\text{t}^3$ or f·t·t·t

This creates an opportunity to discuss various mathematical conventions that involve abbreviations for units.

</td><td>

B

This sequence of tasks:

$9\pi\ \text{cm}^2$
$9\pi\ \text{cm}^3$

provides opportunities for students to make connections between the value of the exponent on the unit and the physical properties of the shape. For instance, you might help students notice that appropriate problems to pose for these two answers could be the area of a circle with radius 3 cm, and the volume of a cylinder with radius 3 cm and height 1 cm. It may be helpful to point out that 9π cm can also be written as $9\pi\ \text{cm}^1$.

</td></tr>
<tr><td>

C

Shapes such as the one in this task, where students cannot readily compute the area of the base, reinforce the use of the forumula:
Volume = (area of the base) × (height)
also written: $V = Bh$.

</td><td>

D

As an extension for item **D**, you can ask students to describe the kind of object that each of the wrong answer choices could refer to.

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E

Students may try to multiply all the given numbers in this problem.
Suggest that students include units throughout computations. This practice, common in chemistry and other sciences, would look like this:
3ft · 4ft · 8ft = 96ft^3

</td></tr>
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F

Students may have trouble recognizing that they have been given the area of the base of the layer, and they may try to compute the area of the base. The goal of this task is that students gain practice with the Volume = (area of base) × (height) strategy, and generalize this for a shape with height n units.

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G & H

Students may need help naming shapes. Manipulatives may be helpful here, as they allow students to move prisms into a more recognizable orientation.

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I

Some students may think $200\pi\ \text{cm}^3$ is correct because it is the area of the base times the height. The goal of this question is to help students attend to the limitations of the $V = Bh$ strategy for finding volume.

</td></tr>
</table>

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Journal of the California Mathematics Project
Volume **7**, 2015

Connecting Concepts in Color: Patterns and Algebra

Shandy Hauk, Sarah Cremer, Cathy Carroll, Katie M. D'Silva,
Mardi Gale, Katie Salguero, and Kimberly Viviani

A mathematical model captures information about the main ideas in a problem situation. The kinds of thinking needed to model with mathematics are central to the Common Core State Standards for Mathematics [CCSSM] (2010). In fact, "model with mathematics" is included in the standards as a valued habit of mind to be cultivated (Standard for Mathematical Practice 4) as well as core content. Modeling includes several kinds of activity: noticing associations among categories and quantities, developing mathematically useful representations of situations, analyzing relationships and representations, taking the risk of starting (and perhaps abandoning) a strategy. Important to modeling are making connections among representations while bringing to mind and choosing strategies for making sense of a scenario (Schifter, Bastable, & Russell, 2009; Seago, Mumme, & Branca, 2004). Clear in the CCSSM and the associated Learning Progression documents is that all of these skills, practices, and habits of mind are sharpened in middle school (Common Core Standards Writing Team, 2015).

We argue that certain uses of color in the mathematical practice of modeling support (a) connections across representations and (b) decision-making among strategies. In particular, students and teachers can use color to keep track of meaning. A common example is writing one equation in red and another in blue and then drawing their graphs in red and blue, respectively. Yet, color can be used for much more.

Our purpose here is pragmatic: to share the evidence-base for using color in certain ways and to describe its use in a particular professional development course. While our examples are with middle grades teachers as learners, the same careful color-anchoring approach can be profoundly effective with middle and high school learners. The math activities included here were part of a U.S. Department of Education-funded research and professional development project (Hauk & Carroll, 2014). After some background and framing for the use of color in the context of patterns and modeling with mathematics, the bulk of what follows is the description of a modeling activity from the professional development course. The activity was part of the first day and was revisited and referenced throughout the year-long experience. We close with a few thoughts on implementing similar activities in professional development and classroom contexts.

Color in Representing Mathematics

When learners relate parts of a mathematical equation to a graph or word problem, or interpret a diagram, they must direct their attention to each individual source, encode separate pieces of information, and then manage the stored information to make meaningful connections. Paying attention to these multiple sources of information can be challenging. Research indicates that the careful use of color-coding and color-matched text and diagrams improves learning (Sweller, 1994). Furthermore, looking at how others incorporate color into their models provides a window into others' thinking (Friel & Markworth, 2009; Smith, Hillen, & Catania, 2007; Toney, Slaten, Peters, & Hauk, 2013).

How learners make (and make sense of) mathematics through the use of color takes two forms. In investigating students' routes from informal mathematical activity to formal mathematical reasoning, Zandieh and Rasmussen (2010) specify a difference between models *of* mathematical activity and models *for* mathematical reasoning. As we illustrate below, incorporating the use of color into classroom or professional development activities includes attention to color in both the representational role as a model *of* a category or quantity and in the strategic role as a model *for* relationships among ideas and actions.

Illustrating the Ideas: Seats at the Table

The *Seats at the Table* professional development activity is based on a classroom lesson originally adapted from the Mathematics Assessment Resource Service [MARS] (1998; see `http://map.mathshell.org/`) for the mathematics support curriculum *Aim for Algebra* (Gale, 2011)

The purpose of the activity as part of the professional development is as a touchstone example for considering the difference between models *of* and models *for*. That is, we use it to distinguish between *multiple representations* and *multiple strategies*. The structure of the professional development activity provides two opportunities to think about the task. First, participants engage in the activity as learners, considering the mathematical content of the task, noting many possible representations for the situation, and narrating and color-coding their thinking. Along the way, they illustrate for each other a variety of possible solution strategies. Later, participants view the task as teachers, considering pedagogical issues and instructional implications. The task, as presented to the participants-as-learners, shows visual and verbal representations for a patterning scenario (see Figure 1). The scenario involves people who are sitting around a line of contiguous trapezoidal tables. Participants were asked to use these representations, and any others they created, to answer the questions:

(1) How many people can sit at 20 tables? 50 tables?
(2) Find a rule to determine the number of seats at any size table.
(3) How do you know your rule works?

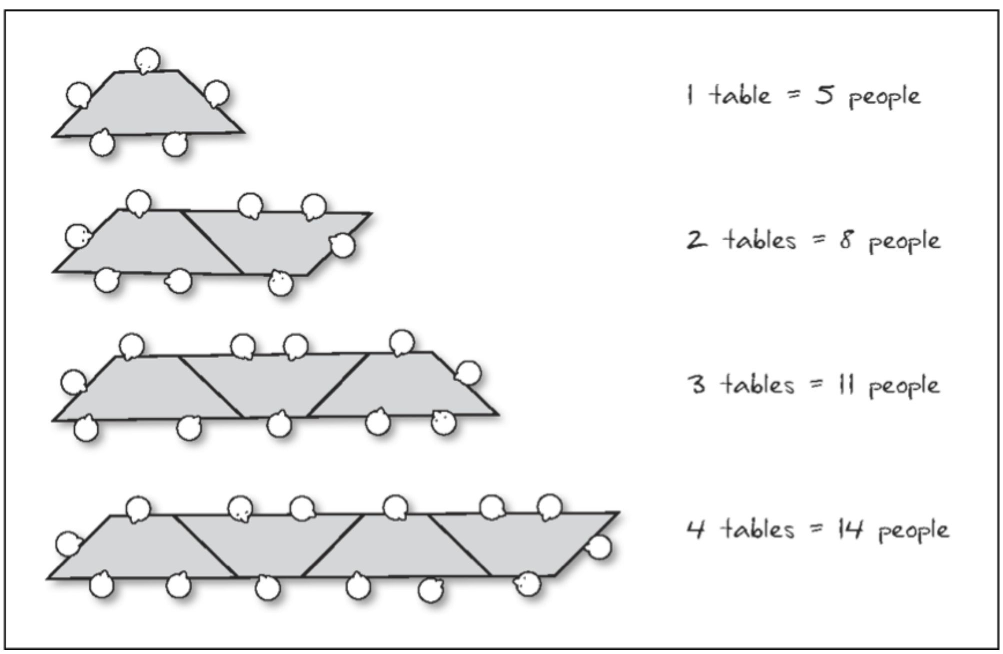
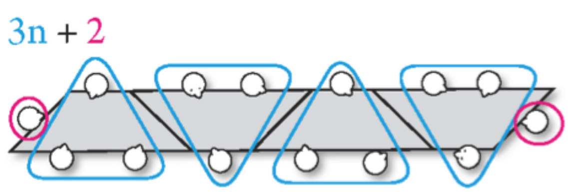


FIGURE 1. Figural representation in the *Seats at the Table activity*

Participants worked individually, and then in small groups. The facilitator monitored the rules that participants developed, then brought everyone together as a whole group to investigate the different rules and how each rule connected to possible visual and tabular representations.

During their work in small groups, participants rarely used color systematically to support their creation of a rule. After 10 minutes of group work, the facilitator introduced the purposeful use of color to connect representations. In particular, she connected the expression $3n + 2$ and the visual of four tables (see Figure 2). As shown, each group of three at the tables is circled in the same color (blue) and the "3n" is written in blue because $3n$ represents 3 people at each of the n tables. The "2" in the expression represents the 2 people at either end of the row of tables, so those two people in the figure are circled with the same color as is used for the "2" in the expression (here, red). The facilitator also generated lists of values using the same colors (shown to the right in Figure 2) to demonstrate the *development* of the expression $3n + 2$ and its connections to the color-coded sketch of people seated at tables. After this demonstration and a short discussion, participants returned to their small groups to apply the idea of systematic use of color to connect their own rule(s) to the visual for four tables.



# of tables	# of seats	How I saw it
1	5	$3(1) + 2$
2	8	$3(2) + 2$
3	11	$3(3) + 2$
4	14	$3(4) + 2$
...	...	...
n	$3n + 2$	$3n + 2$

FIGURE 2. Using color to show the connection between $3n + 2$ and a figural representation.

Figure 3 (next page) includes four examples of how participants used color and an associated symbolic expression. Each uses color-coded chunks of information that allow comparison across models. Learners could compare the ways of thinking about the problem (multiple strategies) and get insight into a variety of mathematically correct models of the pattern (multiple representations).

Each un-simplified expression, or rule, was a window into a person's thinking. The participant who wrote the rule $4 + 3(n - 2) + 4$ explained the scenario as a number of tables of 3 people in the middle, with a table of 4 people at each end. The participant who wrote $5n - 2(n - 1)$ said that each individual table could seat 5 people, but also noted that seats are lost when the line of tables is contiguous. While both are equivalent to the simplified rule $3n + 2$, they express the constant and variable parts in the linear pattern in distinct ways because the sense-making strategies of the two thinkers were distinct.

Mathematical Ideas and Color Assignment

A process of trial and error, including much discussion among project staff and participants in the professional development, honed our recommendations for the use of color. Ultimately, we determined that assigning a color to each term in the expression (and to where that term is "seen" in the visual) led to productive discussions. Participants' sharing and collective sense-making about the justifications for color associations also side-stepped a possibly counter-productive routinizing of the use of color. We wanted to avoid over-simplifications like "always use blue for the term with n in it" to allow for representations like the third and fourth symbolic expressions in Figure 3 (next page). In each case, the variable n appears in two of the terms.

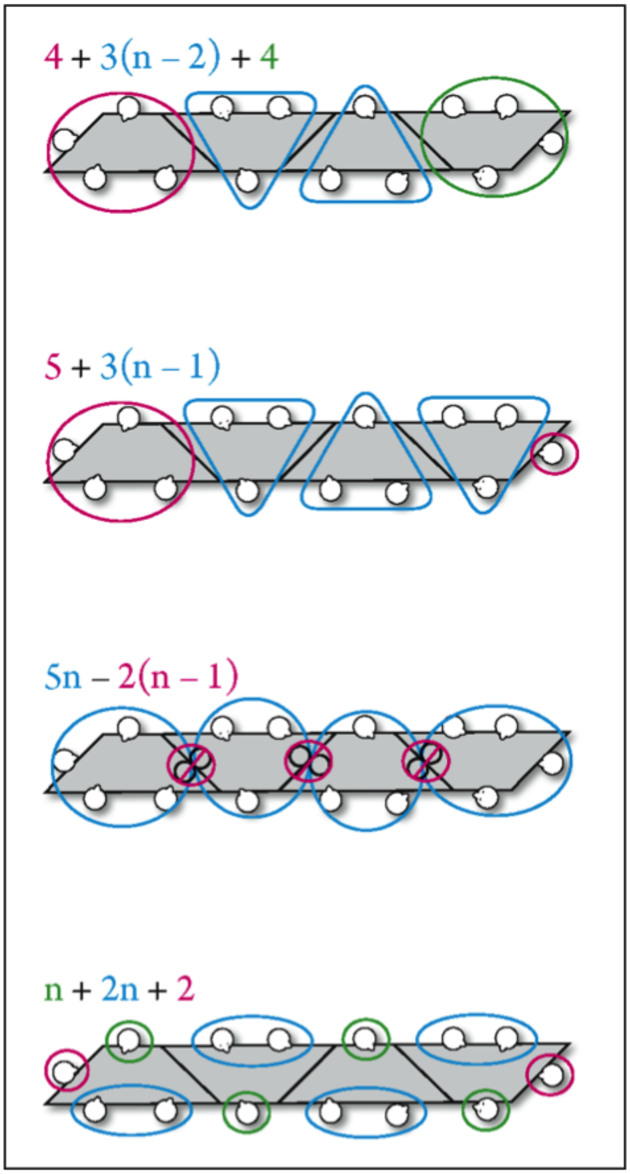


Figure 3. Examples of using color to connect symbolic expressions with figural representation for *Seats at the Table*.

Deciding what constituted a "term" in an expression required comparisons across representations and explanations entailed validation of what the group was going to call a "term" in the symbolic representation of the pattern. That is, reflection on the process of coloring their strategies and discussion among participants helped them decide on the details of a socio-mathematical norm for "term." In particular, it led to nuanced consideration of how the model of a particular characteristic of the figure played a role in describing the model for a strategy. For example, the expression $5n - 2(n-1)$ was a representation that was mathematically equivalent to $n + 2n + 2$, but the color coding and discussion highlighted that it might not be strategically equivalent.

Later, after several tasks that distinguished between the representation(s) of a pattern and the strategies for getting there, participants reconsidered the activity through an instructional lens. They noticed that the elegance of putting a representation in "simplified form" might well obscure valuable information about student thinking processes. This led to conversations about the value of both exploring with representation and practicing with standard forms.

Conclusion

A rich task like *Seats at the Table* allows learners to engage in mathematical practices while also addressing content standards. In the example presented here, we highlighted the practice of modeling with mathematics and focused on the content in the CCSSM Expressions and Equations content domain. Figure 4 summarizes the prerequisite content clusters, target standards, and horizon content clusters (i.e., content that might build on what is addressed in the activity).

Prerequisite standard clusters	
4-OA.C	Generate and analyze patterns.
5-OA.A	Write and interpret numerical expressions.
5-OA.B	Analyze patterns and relationships.
Target standards/clusters	
7-EE.A.2	Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.
7-EE.B	Solve real-life and mathematical problems using numerical and algebraic expressions and equations.
Horizon standard clusters	
A-SSE.A	Interpret the structure of expressions.
A-SSE.B	Write expressions in equivalent forms to solve problems.
A-CED.A	Create equations that describe numbers or relationships.

FIGURE 4. CCSSM Content Standards Alignment for the *Seats at the Table* activity.

When engaged in mathematical modeling, a learner may use a variety of representations: rule, expression, visual, table, and so on. Noticing the similarities and differences in un-simplified models of a pattern (multiple representations provided by multiple thinkers) offers insight into the variety in valid strategies for modeling a scenario. The use of color can highlight connections among elements in the various representations and allow comparison across different strategies. Taking time to examine the different ways of modeling a problem supports teachers and students in building a repertoire of ways to view patterns.

The participants who were part of the professional development agreed that using color to model patterns and algebraic expressions had been very powerful for themselves and for their students. At the end of the year, one participant explained it this way:

> *The use of color with patterns and algebraic equations: that tiny little thing just really transformed my algebra unit this year, and I feel like they understand all those concepts so much better. So I think for them and for me, that's a huge takeaway.*

Though we focused here on the context of professional development, the activity can be scaffolded in the classroom for use with students. Key in the scaffolding are (1) asking learners to make sense of the situation first as individuals and in small groups *before* (2) offering a demonstration of a particular systematic use of color, then (3) follow with time for learners to re-cast their initial efforts using the newly introduced color-coding and (4) whole group time for people to describe how their particular strategy is rendered through color and (5) reflections by learners on things to remember about the process and connections to "simplified" expressions. While other ways for incorporating color into representations could be appropriate, what is important is that the use be consistent across associated representations.

Further evidence about the potential effectiveness of modeling in color as part of professional development and instruction is emerging in early results from a related study of middle school student learning outcomes. These indicate that students in the classes of the project's teacher participants had greater gains in understanding of figural representations, tabular representations, and translation across representations in pattern problems than comparison students in the classes of non-participating teachers.

Acknowledgement

The contents of this report were developed under grant number R305A100454 from the U.S. Department of Education. However, the contents do not necessarily represent the policy of the Department of Education, and you should not assume endorsement by the Federal Government. The findings and conclusions in this report do not necessarily represent the official positions or policies of the funders.

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Journal of the California Mathematics Project
Volume **7**, 2015

Inductive Math Instruction

Michael Sweeley

It is no secret that America no longer leads the world in math achievement. Efforts over the last 15 years, focusing on the integrity of instruction, have assured that every student is taught the correct material, by a qualified teacher, with a standardized textbook bearing the state's seal. Presumably, as the correct materials assembled in the correct fashion will build a sturdy house, solid instruction and instructional materials should produce a solid education. Now, with over a decade of standardized curriculum and assessment behind us, these measures have not borne the results expected. The meager gains in American test scores have been doubled and tripled by Asia and Europe. New classroom equipment aside, we must look beyond the cover, to our approach to education and the underlying philosophy of learning.

Educational policy is the epistemological offspring of Jean Piaget's developmental stage theory, a model charting children's cognitive development through distinct stages of concrete and abstract thought. This gospel of age-appropriate curriculum dictates what should be learned when, that young children should study arithmetic with concrete objects while adolescents enter the abstract realm of algebra. But how does learning happen? What is learning? What physiologically underlies it? For insight into this most basic, yet ineffable topic we turn to the theory of constructivism, modern neurology, and the wisdom of children.

Constructivism, a revolutionary paradigm of learning, asserts that it is the learner who must construct meaning, who is the architect of the knowledge. This theory (and philosophy) was founded by Piaget and reinvented by many, each highlighting different aspects. Focusing on the impetus of learning, Lev Vygotsky emphasized the roles of language and social interaction: meaning is constructed by expressing one's ideas in language and collaborating with other people. In his most divergent and dramatic contribution, Vygotsky also believed that experience informs development. From Vygotsky's perspective, the standard classroom model – the teacher actively disseminating material to passive students as if loading information into a computer – is upside down. Since students must ultimately teach themselves, a teacher can only light the way along their journey. Changing the emphasis from teaching to learning, Vygotsky's work promotes students to center stage, as the star players in their learning.

My first year of high school was met with an unexpected math book: College Preparatory Mathematics (CPM). Twenty years later, I remember learning the Pythagorean Theorem like it was yesterday. This chapter began not with the statement but with a project – an application that in most textbooks would be banished to the back of the chapter, the repository untouched by teachers. The lesson began on the next page with an investigation of right triangles, measuring their sides and recording the data in a table. I noticed that the squares of the shorter sides combine to the square of the longest side. Concluding the investigation, we stated a hypothesis, putting into words the truth unveiled: the Pythagorean Theorem. Embodying Vygotsky's paradigm, the experience preceded the learning. To culminate the chapter, we executed the anticipated project, calculating the length of a guy wire needed run from the corner of a roof to the top of an antenna, putting to use the new tools developed over the chapter. I built a three-dimensional model of the structure described.

Sadly, the novel pedagogical approach was not appreciated by everyone. A favorite subject of parent complaints, CPM was condemned by the board and eventually thrown to the wayside in favor of a more traditional program, the kind the parents wanted, the kind that had not worked for them...Though CPM's life was short at Sonora High School, its spirit lives on in me as a teacher.

CPM's design introduces a topic with specific examples and leads the students to notice patterns and infer general truths. This is an approach I now know as inductive. A typical textbook, in a direct instruction format would announce the Pythagorean Theorem by stating $a^2 + b^2 = c^2$ and follow with specific examples worked out for the student. The jargon of educational theory names many variations of inductive teaching (and inductive learning on the part of the student). Inquiry highlights the questioning aspect used by a teacher. Questioning is a motivating tool and is required of the students as critical thinking in tasks (National Institute – Landmark College, 2005). A lab using the scientific method, requiring students to observe data and form hypotheses, is a quintessential inquiry lesson. Discovery learning provides the student with not the answer but with the means to find it. The learning cycle may be to inductive teaching as the five step lesson plan is to deductive teaching. Inductive teaching, requiring the learner to construct meaning, is inherently constructivist. To classify a lesson as a particular nuance of inductive teaching is not a goal of this article, but most lessons discussed involve inquiry and discovery components. In Vygotsky's vein, cooperative learning often accompanies inductive teaching, as with CPM.

As a graduate teaching assistant, constructivism was not how my teaching began. Like most college instructors, I lectured the entire class period, save a few minutes to answer questions. The students appreciated explicit definitions, numbered theorems, and sequences of steps to rigidly follow presented like street directions. One student said, "Your notes are poetry." I was praised on my evaluation for "thoroughly covering the material." Fortunately, the next educational setting, with a very different educational philosophy, would reformat my teaching.

In my first year of teaching eighth grade algebra at a charter school, I was asked to lecture less. The charter school coordinator enlightened me that eighth graders, though quiet they may be, cannot absorb a 40 minute lecture. Little did I know, starting my second year with more direct instruction than ever that – like a caterpillar entering a cocoon – I would emerge in May a different teacher.

That year blessed me with an unusually heterogeneous group of students. Also, I became very interested in plants. Two seventh graders, having advanced to algebra 1 a quarter into the school year, were always finished early. Not wanting them to feel bored (and cause problems), I let them work ahead in the curriculum and was amazed by what they could do without a lesson. They spent most of the time working together, figuring out the math, helping each other put together the pieces. Coming over to them occasionally, I would answer a question or give them a nudge in the right direction. My guidance was still crucial, but they were in the driver's seat. As these two students became the workers laying down the bricks, I became the inspector overseeing a sound construction.

Soon this seed, a new instructional model, grew into my other classes. I began to group students working on the same topic to rotate around the room, providing short bits of instruction to keep each group moving as they put together the pieces. In one week my classroom had morphed into the learning center model, befitting of a primary classroom, and I had become a CPM teacher. At the beginning of the year, the students answered my questions. At the end of the year, I answered their questions. They were in the driver's seat. The state test scores jumped this year by leaps and bounds. My construct of learning was changing – from that of the teacher building to a new theme of the students growing. I now "tended to" the students, like the new plants in my office, nurturing their growth. Neuroscientists, now peering inside the brain as learning happens, support a plant growth model, in a very literal sense, as the basis of learning.

While direct instruction is still essential, I have grown to favor inductive teaching. I begin my unit on slope and linear equations with boards and tape measures. I place the boards along the wall, a short and a long board at a shallow angle and another pair at a steep angle. I challenge the students with an inquiry: devise a metric, a way to measure the steepness of the boards. The shallow long board and steep short board help them realize that neither the height along the wall nor the distance to the wall along the floor is sufficient; they must find the ratio. The goal to obtain a large number for the steep boards and a small number for the shallow boards, the students are forced to use the rise as the numerator and the run as the denominator. Patterned after CPM's investigation of right triangles, my unit on linear equations begins with graphing different equations: $y = x$, $y = 2x$, $y = 3x$, etc. Then students discover the coefficient as the slope. I then have them predict the effect of the following change: $y = x + 3$, $y = 2x + 3$, $y = 3x + 3$. The students discover the math and tell me the theorem. From specific examples, students make the generalization that the graph of $y = mx + b$ is a line with slope m and $y-$intercept b. This theorem is the students' result. In contrast, a direct instruction approach to this topic would begin with the theorem as the teacher's opening statement and move on to specific examples.

Even after K-12 transformation, I did not think that the condensed pacing at the college would afford any modality other than lecture sessions followed by a few practice problems, or that college students would want a student-centered model. For the last two years at the college, I have gravitated toward an inductive instructional model, reducing my lecture time by more than half. No longer preoccupied with covering every minute detail and case the students will encounter, I leave more for them to construct. Most of my recent classes have achieved much higher test scores, scoring particularly better on problems that require levels of learning higher on Bloom's Taxonomy. Last summer roughly half of my algebra 1 students mastered distance-rate-time and mixture problems, a huge improvement from the past success rate of roughly ten percent. Student evaluations support the change, as summarized in the table below.

Preference	Number of students
Support having less lecture	13
Would have preferred more lecture	3
*Support inquiry-based instruction	4

*The data were limited for these criteria.

Students wrote the following comments:

> *Student 1:* Most people learn by doing, and I think it's crucial to do problems in class. That way you [the teacher] can teach them how to do the problem correctly, versus lecturing, which becomes <u>very</u> <u>boring</u>.

> *Student 2:* I believe it was very beneficial to spend more time in class on problems, as it gives us a chance to give feedback and collaborate with classmates.

My biggest surprise is that inductive teaching has been not only more effective, but also more efficient. For the last two years I have been ahead of my pacing plans and able to cover nearly twice the material. How could this be? First, less time spent on lecture gives students more time to learn. Second, inquiry-based teaching, by using central underlying concepts to drive knowledge and procedures, unifies the curriculum.

By applying the meaning of exponents, my algebra students infer the product rule: five factors of x multiplied by three factors of x reassociates to eight as the power on x. Covering the quotient rule – the next section – in the same lesson, I ask the students to consider $\frac{x^5}{x^3}$, which by the same reasoning leaves two factors of x or x^2. Breaking the tradition of teaching slope-intercept form and point-slope form as two completely different things, I presented them this year as one and the same. Slope intercept form, $y = mx + b$, can be rewritten as $y - b = mx$. I have the students state this equation in words: "The rise equals the run times the slope." Conversely, point-slope form, $y - y_1 = m(x - x_1)$ can be written as $y = m(x - x_1) + y_1$ and be said as "the y value equals the starting y plus the rise" or "the y value equals the starting y plus the product of the run and slope." Furthermore, what we call slope-intercept form is merely the case where $x_1 = 0$. Applying the same concept to quadratic functions, students are able to translate parabolas, a topic normally left to the end of the next course!

In the K-12 system, numerous pieces of research support the effectiveness of inductive instruction. The few schools sticking with CPM have enjoyed great success. Compared to the California state average, 46%

more eighth graders using CPM have scored proficient or advanced on the Algebra 1 CST over the years 2004-2010 (CPM, 2013). A study comparing CPM to traditional instruction found greater test score gains in the CPM group than the comparison (non-CPM) control group (Ferguson, 2010). At the CSU Stanislaus 2013 Math Camp, instructed in an inquiry, project-based format, students improved their test scores by an average of 26% (Sweeley, 2013). Evidence from many studies also suggests that inductive instruction fosters higher retention than traditional instruction. One particularly strong study broke 68 sixth graders into four heterogeneous classes: two with inquiry-based instruction and two controls with traditional instruction. The inquiry-based classes achieved roughly 11% higher retention, the gains increasing over time (Serrano, 2012).

Students who learn math inductively are also more likely to adapt their skills to different problems and apply concepts to different contexts (Ferguson, 2010). Driven by conceptual understanding rather than rote memorization, inductive learning surpasses direct instruction in retention and flexibility. Other research suggests that, through inductive learning, students also acquire critical thinking skills and that these broader skills can be transferred to other content areas (Bangert-Drowns & Bankert, 1990). The philosophy of inductive learning, particularly Vygotsky's brand of social constructivism, concurs with what is achieved through doing, discussing, and reciprocal teaching – not by being lectured at (Dunlosky, Rawson, Marsh, Nathan, & Willingham, 2013). Additional research has offered information about how writing improves learning of mathematical content (Bagley & Gallenberger, 1992) and tutoring improves long-term retention (Fitch & Semb, 1993).

While inquiry-based instruction arguably offers more powerful learning than direct instruction, it must be deftly executed. My classroom experience and student comments concur with Dr. Sallee's reported research on the following points (Sallee, 2003, 2013).

(1) Inductive instruction, while student-centered, does not diminish the role of the teacher. First, planning inductive instruction requires much more thought and time. Second while the teacher tells less information directly, the teacher's new focus, as the guide of conceptual architecture, is to ask questions that lead students in the right direction. Third, explorative activities require excellent directions; in this area the author has adopted more thorough and explicit instruction. Compared to direct instruction, the teacher's role in inductive instruction is lesser in providing the content but greater in providing the learning experience. In addition, the teacher's time not spent in front of the class is usually spent moving around the room, working with groups or individual students.
(2) After an exploration, it is crucial to discuss and apply the learning. The learners should describe the data they collected, their strategies, and the ideas they formed. To help the students clarify concepts, a teacher can either ask additional questions or resort to direct instruction. Application, integral to the students' learning, also serves the teacher as a vehicle of assessment.
(3) Inquiry-based instruction does not dispense with direct instruction. Some direct instruction is always needed, and the two instructional models are not mutually exclusive. Direct instruction often follows an exploration, and can serve as Plan B for students who did not attain the desired result.

In retrospect, if my introduction to the Pythagorean Theorem had been the abstruse statement, I would not have appreciated it as much, perhaps not even accepted it. Like a child proudly holding a just constructed Lego model, students take greater ownership of mathematics when they help to build it. A tribute to Vygotsky, teachers should remember that in addition to sharing content knowledge, they are helping young people to build their minds.

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Journal of the California Mathematics Project
Volume **7**, 2015

Remark: This article is offered as a "professional practice" piece. It is an example of what an external reviewer report might look like when a department completes a multi-year reflective assessment of its programs. Names of people and places have been changed and some details modified to protect the anonymity of the original department.

Alpha University Department of Mathematics Five-Year Program Review (May 2010 to May 2014)

Viji K. Sundar

As I begin this report as the external reviewer of Alpha University, I want to express my sincere thanks and appreciation to Dr. Graph and to the Self-Study team for providing a document that is very detailed and comprehensive in the data and analyses needed for assessing the program. Dr. Graph's availability to respond to my calls and emails made this task a joy.

1. Introduction

The mission of Alpha University is to make lifelong learning opportunities accessible, challenging, and relevant to a diverse student population. The core values that shape the mission are quality, differentiation, relevance, affordability, and community. Alpha University's central purpose is to promote continuous learning by offering a diversity of instructional approaches, by encouraging scholarship, by engaging in collaborative community service, and by empowering its constituents to become responsible citizens in an interdependent, pluralistic, global community.

The mission of the Alpha University Department of Mathematics is to provide an outstanding educational experience to undergraduate and graduate students. The degree programs emphasize the development of critical thinking, quantitative skills, and academic and professional integrity to investigate the natural world. The department strives to train students to participate as global citizens with strong communicative, analytical and scientific abilities, while being able to think and cooperate across conceptual disciplines.

The details of the reviewer's findings below will show that there is a good fit between the University's mission and the Department's mission. An online option for the Bachelor of Science Degree in Mathematics was approved in 2013. As of Summer 2015, the Department terminated all on-site offerings of mathematics courses and launched the online Bachelor of Science Degree in Mathematics. The mathematics department's move to an online program merges with its mission "to prepare students to participate as global citizens . . . " and the University's mission "to provide an inclusive and accessible environment . . . supporting students, staff, faculty, and administrators in their work as responsible participants in society."

This report is based on the documentation provided by the Department and a 90-minute interview with mathematics faculty and some students.

2. Strengths

The documentation I received from the Department was thorough. It included:

- Guidelines for Five-Year Program Review, which enabled the reviewer to navigate through the implementation of the self-study;
- Catalog Descriptions of each of the five year's BS in Mathematics program which enabled the reviewer to assess the growth/changes in the program;
- Program Learning Outcomes – not only for the program as a whole but also for all the courses in the catalog, which made the assessment transparent;
- Curriculum Map that is very well laid out in a student friendly table format;
- Multi-Year Assessment Plan (2010-2014) with a matrix for each of the five years indicating the Measure Instrument and Program Learning Outcomes (PLOs);
- The five-year self study plan and the detailed implementation time line (with who/what/when), and the reflective questions and responses;
- The final set of documentation on the entire self-study is quite exhaustive as it included the ten requirements of a self-study.

This is impressive because, only very few campuses do the self-study as methodically and as completely as the Alpha University Department of Mathematics for each year. This self-study committee's Chair had a plan and implemented it decisively and well.

The Mathematics Curriculum

The documentation provided for the Bachelor of Science (BS) in Mathematics has changed since the last review. There is no longer a BS degree in Applied Mathematics. That is welcome to this reviewer as that option – more often than not – tends to take away the theory which is the core of the discipline. On the contrary, the revision has resulted in a much stronger curriculum in both areas, BS in mathematics and BS in mathematics with Single-Subject Mathematics Credential Program (SMCP). The current preparation for the BS in mathematics major has added the courses Numerical Analysis, Abstract Algebra, Functions of Complex Variables, a Mathematics Project Course, Problem Solving Strategies and Methods of Teaching Math. As such, a math major opting for SMCP enters the SMCP subprogram with a stronger foundation of mathematical content. In order to complete the SMCP requirements, the student needs only two additional courses: Student Teaching with Portfolio, and Technology in Math Teaching.

The documentation provided shows a very good alignment of the Department's curriculum and the California Common Core State Standards. Mathematics majors need to be alerted to the connection between the abstract concepts they learn in upper division courses and the secondary school mathematics. The course descriptions provided has several examples of this. For example in the Algebraic Structures course students will learn how homeomorphisms are involved in solving a polynomial equation; in Foundations of Geometry will discuss how non-Euclidean Geometry attempts to describe the physical nature of the universe; in History of Mathematics the students examine how development of mathematical ideas charge the society (e.g. Hindu-Arabic numerals forever altered commerce, navigation and surveying); and in Abstract Algebra the base of coding theory as an application of finite fields.

Program Learning Outcomes

While many campuses across the nation are articulating Program Learning Outcomes (PLOs), the Department has included this in the University catalog. The PLOs have been revised since the last review, not so much in the essence but to make them understandable to prospective students. The reviewer would like to include the PLOs and comment on their merit as the collection of PLOs meets different agencies requirements. The PLOs are indicators of the way the delivery of instruction meets the Standards of Mathematical Practice, which in turn, shows that the BS in Mathematics program complies with the California Common Core Standards.

3. Recommendations

(1) The Department should consider revising the course Methods of Teaching Math in the requirements for the major. Although the course name and course descriptions are appropriate for SMCP program, and the art of "teaching and communicating" is much needed universal skill, it may not be thus understood by a future employer in a corporation.
(2) The Department should consider renaming the courseTechnology in Math Education to something like Technology in Mathematical Sciences and make it a requirement for BS in mathematics.
(3) The reviewer recommends that Advanced Calculus be renamed Real Analysis. The classical and yet fundamental characteristics of Real Numbers - Cantor's Theorem, Heine-Borel Theorem, and Lebesgue Measure to name a few - are subsumed in this course.
(4) The reviewer recommends that the administration provide additional resources to the Department in two areas: (a) provide funds to hire more tutors so students do not have to wait up to a week to get a question answered and (b) provide funds to purchase the needed hardware and special software that directly impacts on the teaching and the student learning. When a Department offers"online courses" in a program it is only prudent to get all the needed technology to make this accessible and inviting. The reviewer is surprised that an "online" program is not equipped to access modern technology.
(5) The reviewer was able to find documentation of Department?s commitment to critical thinking, writing, and communication incorporated in several courses.The reviewer did not see the diversity component or any reference to equity. The reviewer strongly recommends that equity be incorporated. Additional information is available at `http://www.todos-math.org`. The mission of *TODOS: Mathematics for ALL* is to advocate for equity and high quality mathematics education for all students – in particular, Latina/o students.

4. Conclusion

Alpha University's five-year self study report and the interaction the reviewer had with the faculty and students presents the picture of a Department with a BS Degree program that is robust in its curriculum, faculty who are well respected and admired by their students for their teaching style, accessibility and caring. The reviewer was pleasantly surprised at the interaction between faculty and students in an online program. Besides students across the State, the program has a handful of students from abroad who are satisfied with the classes and the format of delivery. There is remarkable move on the part of the campus to "bring the college classroom" to the student. The next move may be to expand the program globally by recruiting teaching faculty from abroad. There is a great possibility both in offerings of courses and the recruiting of students/faculty in the "online" BS in mathematics program.

Addendum to the above Report

The reviewer visited the Alpha University campus. During this visit the reviewer had a face-to-face meeting with Dean Tremblay, Chair Martinez and Professor Graph at Dean Tremblay's Office. The Alpha University team acknowledged that the review was thorough and the recommendations timely and to the point. Dean Tremblay assured the Mathematics Department would be given sufficient funds and resources to follow through with the recommendations. The next step would be for Chair Martinez and Professor Graph to take this to the math department faculty and implement the suggested recommendations as needed.

About the Author

Viji Sundar ⟨vsundar@csustan.edu⟩ is a professor of mathematics at California State University, Stanislaus and is Director of the Central California Mathematics Project. Her professional interests include the preparation and development of mathematics teachers, innovations in teaching and learning in primary, secondary, and post-secondary schools, and lively conversation about these both in print and in person.

Journal of the California Mathematics Project
Volume **7**, 2015

GEOMETRY LESSON
Flying Kites: Raising Students Sky High

Veronica C. Chaidez and Jessica Romo

Overview

Concepts: Geometry, Measurement, Number Sense, Mathematical Reasoning, Critical Thinking, Number and Operations, Communication

Skills: Identify the properties of a kite and a rhombus, create a kite or a rhombus blueprint aligned with the geometric properties of each individual geometric shape, construct the kite or the rhombus to scale using the previously designed blueprint.

Vocabulary: *rhombus* (a quadrilateral with all sides equal in length; a square is a special case of rhombus); *kite* (a quadrilateral with two distinct pairs of adjacent sides; a rhombus is a special case of kite); *midpoint, vertex, vertices.*

Properties: *rhombus:* Any side can be a base; the altitude of a rhombus is the perpendicular distance from the base to the opposite side (or, when needed, its extension); the area of a rhombus can be found in several ways (see bonus activities, below). *kite:* At least one pair of opposite angles have equal measure; angles between unequal sides have equal measure; no angle is greater than 180° (a kite becomes a *dart* when one of the unpaired angles is greater than 180°).

Grade Level(s): 6, 7, 8.

Duration/Length: Three (3) 50 minute periods.

Prior Knowledge: measuring using a ruler, scaling measurement with ratio or proportion, right angles, and attributes of line segments and shapes (e.g., diagonals, midpoint, vertex).

California Common Core State Standards: Grade 6: 6.RP.1, 6.RP.3.d, 6.NS.3; Grade 7: 7.RP.2.b, 7.NS.2.a, 7.NS.3, 7.G.1, 7.G.2; Grade 8: 8.G.1.3, 8.G.1.4. California website for the state standards: `www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf`.

Materials: 8.5”× 11” white copier or grid line paper, markers, color crayons, color pencils, rulers, protractor, markers, light colored (e.g., white) 13-gallon plastic trash bags, electrical tape, dowels 1/8” × 36” (two for each group), super twine string, permanent markers, and scissors.

Background: What makes a kite fly? What are the mathematical concepts addressed in the construction of a kite? The goal of this project is to have students develop, construct, and fly a handmade kite. By the end of the project, not only will students recognize that the three main forces that affect the flight of a kite (lift, gravity, and drag) are crucial but also that a kite's geometry is integral to its balance and flight capabilities. Through this engaging, hands-on project, students will apply critical thinking skills and spatial reasoning and deepen their understanding of measurement, geometric constructions, scale, and proportion. This lesson can be used to review geometry content or to introduce new material, including academic language.

Day 1

Description: The lesson begins with a teacher-led discussion that will ultimately lead to a student introductory activity on the development of a flying kite.

Materials: 8.5" × 11" plain paper or grid paper, ruler, and colored pencils or crayons.

Teacher Activities	**Student Activities**
Step 1 (15 minutes) Launch. Teacher starts with class in a discussion exploring the properties of kites and rhombuses. While displaying illustrations of each quadrilateral, the teacher asks students to identify some visible attributes of each quadrilateral. The teacher guides class discussion toward discovery of the properties unique to a kite and a rhombus. The classroom discourse goal is to simultaneously construct curriculum knowledge and develop academic language associated with the mathematical concepts. To support language development, vocabulary terms are introduced or reviewed in tandem with notation symbols (e.g., tick marks symbolizing congruent segments). Ensure discussion addresses line segments in polygons (i.e., perpendicular and intersecting lines). On the word wall, whiteboard, or poster board teacher gathers the identified characteristics and academic language throughout the discussion for students to refer back to as needed.	The students participate in a class discussion, pinpointing some of the distinguishable attributes of the two quadrilaterals. This will allow them to discover attributes that are specific to each shape. Furthermore, they will be able to see any similarities that both shapes may have. As the students are identifying characteristics of the shapes, they are asked to use the academic terminology associated with them. This allows students to become familiar with the terminology that they will be using for the subsequent lessons. By the end of the class discussion, students will be familiar with the properties of kites and rhombuses.
Step 2 (30 minutes) Hands-on student activity. The teacher strategically puts students in groups of 2 to 4 members. Each group's task is to create scaled drawings of two distinct flying shape designs (kite and rhombus) on 8.5" × 11" paper, highlighting the properties of each quadrilateral. Depending on the grade level, the scale could be provided by the teacher, or it can be created by the students. Once scale drawings are completed, within the group students compare and contrast the two designs and form a hypothesis and justification about whether one design will fly better or that both will perform equally well. Students enter the conjecture and rationale into their individual notebooks.	Once the students have become familiar with the properties related to each figure, they work in assigned groups of 2 to 4 members. Each group creates a scale drawing for each of the two designs: a true rhombus and a true kite. Each group will need to discuss which of the two designs they predict will function the best when flying and why. Each student will enter this hypothesis into her or his notebook, along with deductive rationale, based on the properties of each quadrilateral. At the end of their group discussion, the members must agree on one design to test, which will they will create the following day.
Step 3 (5 minutes) Whole class discussion. The teacher poses questions with the intent of reinforcing what the students have learned. Select students to purposefully scaffold discussion (e.g., see Stein, Engle, Smith, & Hughes, 2008). Closure Questions: • What is at least one academic or mathematical word you learned today? • What mathematics did we do today? Explain as if you were describing the lesson to someone who was absent. • What was the most challenging concept? Explain.	Students participate in the closure discussion. As the students are responding, they will be reinforcing the material that they just learned and clarifying misconceptions that were left unaddressed in the beginning of the lesson.

Day 2

Description: The lesson will begin with a teacher-led discussion that will review and reflect on content from the previous lesson. After the class discussion, each individual group will engage in the construction of their pre-decided flying shape design.

Materials: light colored/white plastic trash bags, electrical tape, dowels, string, different colors of permanent markers (e.g., *Sharpies*), scissors, and rulers.

Teacher Activities	Student Activities
Step 1 (5 minutes) Launch. Begin the lesson by reviewing the properties of a kite and a rhombus in the form of a teacher-guided class discussion in which every student has the opportunity to share information and ideas. Throughout the discussion, the teacher assesses student content knowledge and determines if further review is needed. The teacher answers questions and addresses any related student concerns before starting the group activity.	The lesson begins with students participating in a class discussion. Students have the opportunity to review the material from the previous day: properties of kites and rhombuses. They also have the opportunity to ask questions about the content or the instructions for the group activity that will follow.
Step 2 (25 minutes) Construct the flying shape. Students get in their pre-assigned groups and appoint a "runner" to collect materials. *Note: Post the materials list in an area that is visible to all students.* As student groups create and decorate their flying shapes, the teacher walks the classroom, making sure students are on task and following the directions correctly (see Instruction Sheet, page 15). If it seems many groups are struggling, the teacher can bring the class together and review necessary steps with the entire class. Student groups follow their scale drawing "blueprint" to construct a flying shape, in the form of a kite or rhombus. Keep in mind that the student activity sheet has them use the doubled thickness of the bag and one-half of the outline to mark and cut a mirrored half-shape that will be unfolded to create the full shape. *Suggestion: Have groups personalize their kites using colored markers.*	Students return to Day 1 groups and select a "runner" who gathers necessary materials to construct one flying shape. Students draw an outline on the garbage bag, cut the bag along the outline, unfold, and verify the scaled measurements of the diagonals. The lengths of the diagonals are used to measure and cut the dowel sticks to the needed sizes. Students tie the dowel sticks together with string and firmly attach the four dowel ends to the four corners of the cut plastic bag using electrical tape. Lastly, they attach a long piece of string to their kites at the intersection of the dowel sticks.
Step 3 (12 minutes). Flying! The class goes to an open space where they can easily attempt to fly their shapes. The teacher makes sure that each group member gets a chance to fly the group's shape. Additionally, the teacher ensures groups jot down key observations about trying to fly the shape. The teacher also encourages groups to trade with trade with at least one other group, preferably with a group that chose the alternate quadrilateral, and repeat the flying attempt and making of observations and notes.	Each group member has the opportunity to get the shape flying. Once they have tried their own shape, students trade with at least one other group, preferably with a group that chose the alternate geometric shape. Students record observations in their notebooks (e.g., challenges that arose while flying each shape; comparing and contrasting the experience of flying different shapes).
Step 4 (8 minutes) Debrief. All return to the classroom for a short discussion. The teacher asks questions regarding student observations and challenges. The teacher also asks students for a discussion that compares and contrasts the two types of shapes when flown. The teacher leaves the discussion open-ended in order to promote an authentic reflection from the students in the reflection assignment (see below).	Students discuss their individual experiences, sharing similarities and differences identified between flying the two different kinds of shape. The students share and compare results with the entire class, making notes for use in the homework reflection and for the next class meeting.

Assignment/Reflection Paper (homework): The students are instructed to write a reflection paper describing whether or not the empirical results of the experiment support their theoretically-grounded hypotheses. They must analyze data collected during Step 3 but may also include other experience. The purpose of the reflection is for the students to analyze their data, draw conclusion, and support their claim through cited evidence.

Reflection Question: Did the shapes fly the same? If not, which type of shape flew the best (kite or rhombus)? Why?

Day 3

Description: The lesson begins with a teacher-led discussion that reviews and reflects on content from the previous lesson. After the class discussion, each student group creates a presentation on the flying shapes experience and mathematical concepts.

Materials: Notes from previous days, access to presentation materials or software (e.g., poster board or Google Doc presentation), flying shapes from Day 2 (optional).

Teacher Activities	**Student Activities**
Step 1 (5 minutes) Launch. The teacher begins the lesson with a guided discussion regarding student individual reflection papers and their final conjectures on whether the shapes flew the same or if one flew more effectively. The teacher scaffolds discussion by seeking purposefully chosen volunteers to start with commonly noted ideas and increase complexity to more subtle aspects of the similarities and differences in the shapes and their flying performance.	The students engage in the class discussion regarding their conjectures. They share key ideas supporting their claims.
Step 2 (30 minutes) Presentation preparation. The teacher directs students to assemble into their pre-assigned groups and to (1) give each person up to 2 minutes to state the hypothesis and support from their reflection paper and (2) after all ideas are presented, discuss their reflections within the group. Then, (3) students must come to a consensus, from their discussion, on whether the shapes flew with the same qualities or if one was more effective than the other. Once the students come to a consensus, they (4) brainstorm an outline for their presentation and (5) as a group create a short Google Docs presentation on their final findings. As the students work, the teacher circulates the room, assessing progress and assisting students as needed.	The students get into their groups. Each group member is required to share their conjectures with supporting information. Once each group member has shared, the group as a whole reviews ideas and comes to a consensual hypothesis that either one quadrilateral design performs better as a flying shape or the two are equivalent. Next, they develop an outline for their presentation, which includes their hypothesis and supporting evidence (e.g., personal experience, specific mechanics of the flying shape, data from class discussions, and photos or illustrations).
Step 3 (10 minutes) Share presentation. As groups finish, pair them to share their presentation with another group. As the students work, the teacher circulates the room, assessing progress and assisting students as needed. *Note: The lesson could be extended for one more day in order to allow all the groups to present their findings to the entire class instead of just one other group.*	Once students complete their presentation (e.g., in Google Docs), they share the presentation with another group and are audience to the other group's presentation.
Step 4 (5 minutes) Closure. The teacher brings the class together as a whole for one last class discussion revealing that the rhombus design was a challenge to fly due to the highly regular geometric properties of a rhombus being inferior in flight.	The students join the class discussion and offer ongoing commentary during the discussion.

Instruction Sheet for Creating the Flying Shape

(1) Unfold or unroll the garbage bag so it lies flat, but DO NOT OPEN the bag. To ensure a large, even, shape, identify a folded edge of the bag (no seam) and consider how to draw the shape to use as much of the bag surface as possible. Keep in mind that after the outer edges are cut through both layers of the bag, it unfolds into the desired shape.
(2) Using your scale drawing for reference, align the longest diagonal along the folded edge of the bag then measure and draw an outline of HALF of the chosen shape (kite or rhombus) onto the garbage bag.
(3) Cut the outer edges of the shape through both layers of the bag.
(4) After the shape has been cut, unfold the garbage bag. At this point, the shape will unfold in its entirety.
(5) Measure the diagonals of the shape in order to accurately measure and cut two dowel sticks, one for each diagonal. Once the sticks have been cut to the appropriate lengths, tie the two dowel sticks together (for a rhombus, at their midpoints; for a kite the dowels cross at a ninety degree angle and are tied at the midpoint of the short diagonal and wherever is needed on the long diagonal to make the ends of the sticks line up with the vertices). After tying the sticks together, place the sticks on top of the shape, making sure that the ends of the sticks are at the vertices. Once the sticks have been placed, securely tape the ends to the plastic bag vertices using the electrical tape.
(6) Cut a piece of string about 4 yards long. Tie one end of this long string around the intersection between the sticks. This will help secure the sticks together and also serve as the tether for flying the shape.
(7) At this point, the shape is complete and ready to fly. Personalize the flying shape by drawing on the plastic with permanent markers.

Bonus Extension Questions

Bonus 1. There are several ways to find the area of a rhombus. Two correct formulas are given below. Based on what you know about finding the area of rectangles and of triangles, explain why each formula works. Include drawings to illustrate how the two formulas are geometrically equivalent.

$$\textit{Area of a rhombus} = b \times a, \text{ where } b \text{ is the length of the base and } a \text{ is the length of the altitude (height).}$$

$$\textit{Area of a rhombus} = \frac{d \times D}{2}, \text{where } d \text{ is the length of one diagonal and } D \text{ is the length of the other diagonal.}$$

Bonus 2. A rhombus is a special kind of kite. Do the same formulas work for finding the area of a kite? Why/why not?

Acknowledgement

We would like to thank Dr. Viji Sundar from CSU Stanislaus and the Central California Mathematics Project for support in making this project possible.

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Journal of the California Mathematics Project
Volume **7**, 2015

GEOMETRY LESSON
When Will I Ever Use It? Putt-Putt Golf

Maria Figueroa and Fabian Jauregui

Overview

Concepts: Geometry, Shapes in two- and three-dimensions, Lines, Angles, Mathematical Reasoning, Communication

Skills: Identify the properties of three dimensional interaction, scale, and design for life-sized putt-putt golf.

Vocabulary: *3-D net* (flattened model that, when folded, creates a three dimensional shape); *cube, pentagonal, pyramid, face, vertex, vertices.*

Grade Level(s): 9, 10, 11, 12.

Duration/Length: 150 to 200 minutes (e.g., three to four 50-minute periods).

Prior Knowledge: measuring using a ruler and protractor, scaling measurements, attributes of line segments and shapes on flat and non-flat surfaces in 2 and 3-dimensions.

California Common Core State Standards:
CCSS.MATH.CONTENT.HSG.MG.A.1: Using geometric shapes, measures, and properties to describe objects.
CCSS.MATH.CONTENT.HSG.MG.A.3: Applying geometric methods to solve design problems.

Materials: Cardboard, tape, scissors, safety compasses, rulers, printer paper, golf balls, putting sticks, cups, markers, printouts of nets for 3-D shapes.

Preparation: Pre-cut pieces of cardboard in order to model assembling a putting hole. At least one pre-made putting hole to demonstrate how the golf course will work.

Background: The real world is math. Often times, mathematics transpires in places we do not consider and can take for granted. Looking back, many people can recall memories of attending an amusement park. Playing putt-putt or miniature golf is a focal point of such a venue. Why? Because it is fun! It is challenging! There is an attraction of striking a ball through a miniature obstacle course that entices people to play. However, does anyone ever think of the work behind creating such an alluring obstacle course? What kind of design and engineering goes into building these collections of blockades and barriers? Think about all the geometry that might be used in creating a putting hole. First and foremost, angles come to mind. The more challenging putting holes are typically accompanied by angles, which range from acute to obtuse. In addition to these angles, putting holes are complemented by lines. The lines serve the purpose of guiding the ball in the desired path. Next, we begin to think about the bumpers: those treacherous and conniving bumpers that are implemented only to obstruct our passageways to success. Bumpers come in all shapes and forms. So, what three-dimensional shapes can someone put into their golf course to make it more appealing? Upon gathering our thoughts, we begin to recognize the mathematical principles that surround a common activity such as putt-putt. Through remembrance, understanding, analysis, and creation we will be able to construct a golf course?a golf course brought about by math into the real world.

Description

This project should take place after the students have knowledge of the different types of lines, angles, and three-dimensional shapes along with all of their properties. This project is intended to serve as a class project that encompasses all of their knowledge together.

Teacher Activities	**Student Activities**
Step 1 (10 – 15 minutes) Launch. To begin, the teacher displays examples to the students ranging from basic courses to elaborate constructions. The purpose is to show the students what they will create by the end of the lesson	The students observe the examples and begin to brainstorm their ideas. They should get an idea of what they like from the examples and think of ways that they can personalize their own putting hole.
Step 2 (8 – 10 minutes) Hands-on with a model. The teacher exhibits a model of a putting-hole for class interaction. This serves as a hook to get the students engaged in the lesson by sparking their interest in the project.	The students interact with the putting-hole model(s) and analyze construction. This way, the students get a first-hand experience of how their own putting hole might need to function. In addition, students observe how many attempts it takes them to score. So, they will be able to assess some of the challenges that occur and build on the experience.
Step 3 (15 minutes) Guidelines and materials. The teacher gives students the requirements for their putting holes. Each putting hole model must: (1) incorporate a minimum of 4 different shapes, (2) use at least one pair of lines (parallel, perpendicular, intersecting). The teacher hands out the printouts of the nets to the students (see page 40). These nets will serve as outlines so that students can create their shapes out of cardboard. As such, the students will have an opportunity to see the different shapes that they can make so that they can decide on what works best for their own putting hole.	After seeing requirements for the project, students are placed in groups of 3 to 4. Each group creates a "blueprint" of their contribution to the putt-putt golf course. They brainstorm the construction of their putting holes, implement their desired level of creativity, and display to the teacher that they have met all necessary requirements before construction begins. Supply groups with printer paper, rulers, and safety compasses to create their single putting hole blueprint. Each group finishes their design blueprint and shows to the teacher before starting to build the actual putt-putt hole. This will ensure that the students are on the right path.
Step 4 (90 – 120 minutes) Construction. Students work with their groups to create parts and assemble the putting holes. The teacher will supply the students with cardboard, scissors, tape, rulers, safety compasses, and cups. The students should be supplied with a long piece of cardboard to serve as the base of their putting hole. This will make their putting hole moveable and they will be able to use it on any surface such as grass, dirt, or cement. Supply students with additional cardboard to create three-dimensional shapes and any walls or bumpers that they may want to include. Also, it is important to note that the cups are intended to	Groups should troubleshoot their putting holes along the way so that they can find any imperfections that need to be fixed and make any necessary modifications. The cup must be positioned so that it allows players to remove the ball after they have scored. The teacher will have a few golf balls and putting sticks available for the students to test their work. The teacher will provide the students with assistance if needed.

serve as the target of putting but are not limited to the one requirement (e.g., a cup on its side may serve as a bumper if the ball comes from one direction but as the hole if the ball comes from another direction).	
Step 5 (30 – 40 minutes) The whole course. The class assembles all of the holes together to create a large course that includes everyone's putting holes. Pairs of students play through the course (pairs start two holes apart). Let students know about the assignment (see below) so they can make notes on easiest, most challenging, and most creative putting holes as they play. The students should be supplied with their own putting stick and golf ball (if possible).	At the end of the lesson, after interacting with all the putting holes, the students will write a short paper (see below). For doing this, the students will have to go to each hole, analyze it, and remark on each other's constructions.

Assignment/Reflection Paper: Each student writes an analysis describing which hole seemed easiest and why, which seemed hardest and why, and which seemed most creative and in what way. They must refer to data collected during Step 5 but may also include other experience. The purpose of the reflection is for students to draw conclusions and support their claims through cited evidence.

Extension: Depending on the grade level, the sheer number of requirements will vary. For example, 12th grade students will have to complete more requirements in order to make their golf courses more elaborate and rigorous whereas 9th grade students will have to implement the basic requirements. Furthermore, the teacher may require the students to calculate the volume of their three-dimensional shapes to optimize the space of their golf course. Another suggestion is to have the students calculate the average amount of tries to complete the hole. This can extend to a statistical approach as well. Depending on the amount of empirical evidence collected in Step 5, the students may be able to calculate the odds of making a hole-in-one.

Acknowledgement

We would like to thank Dr. Donna Andrews from CSU Stanislaus for sparking this idea. We would also like to thank Dr. Viji Sundar from CSU Stanislaus and the Central California Mathematics Project for their support in making this project possible.

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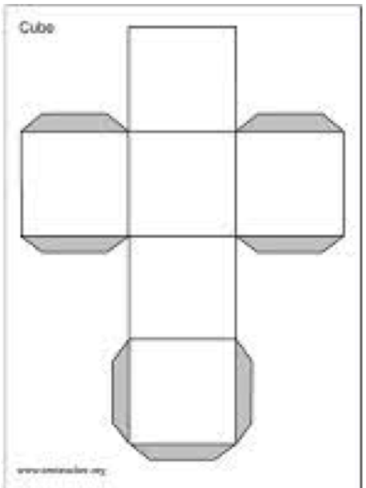
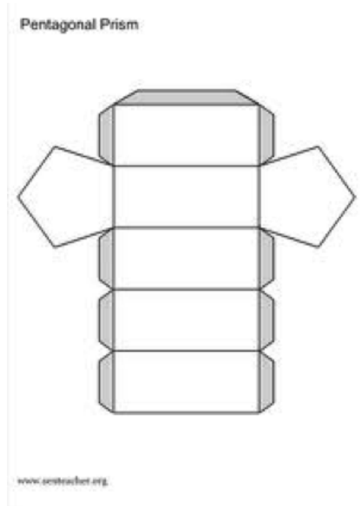
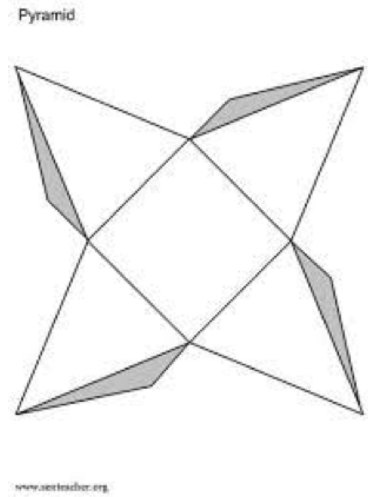
Sample Nets for Putt-Putt Golf Hole Shapes







Two More Examples

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California Mathematics Project

Volume 7, Summer 2015