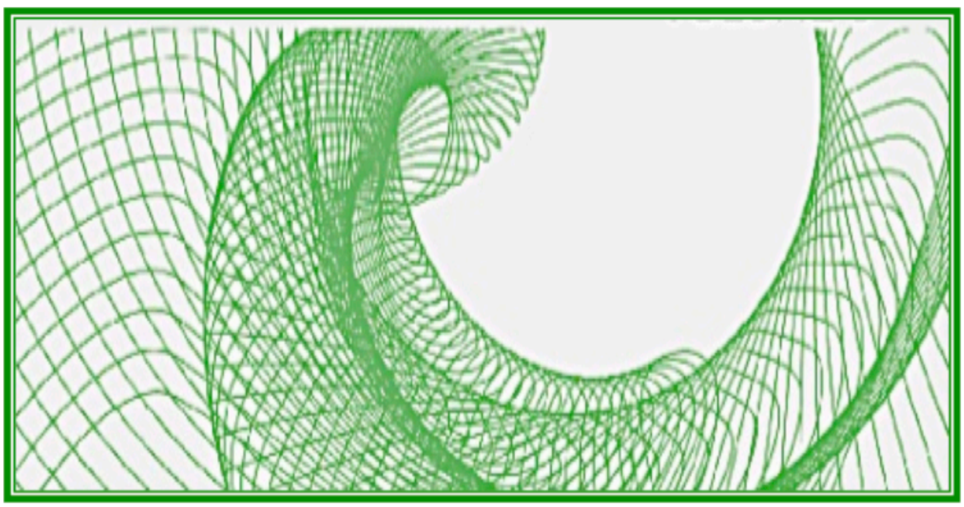
JOURNAL OF THE CALIFORNIA MATHEMATICS PROJECT



CALIFORNIA STATE UNIVERSITY STANISLAUS

VOLUME 6 SUMMER 2013

Introduction

The Journal of the California Mathematics Project (JCMP) is a supplementary publication of the California Mathematics Project (CMP) and is designed to publish articles by teachers to help teachers: "Here is what works for me." The JCMP is a publication sponsored by CSU Stanislaus. The official publication of the CMP is the California Online Mathematics Education Times (COMET), an electronic journal published by Professor Carol Fry Bohlin at CSU Fresno.

Teachers who find methodologies that work well for them in the classroom are encouraged to share these ideas here.

Submitting an Article

Manuscripts should be typed single spaced with a margins of 1.25 inches and a right margin of 1 inch. Any word processor may be used but we prefer articles written in LaTeX. Send manuscripts to:

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Preface

In this Summer 2013 issue the reader will find articles that reflect the mission of this journal.

The article by Allison Toney, et al., on the use of color in teaching geometry seems like a novel idea, yet, it was put forth by Oliver Byrne in 1847. Toney, et al., however, examine the efficacy of using color to teach geometry through the lens of modern research and offer some qualitative data.

On teacher training, Diane Lau reminds us of the *basics* of good teaching and shares some of the ideas she uses in her mathematics teacher training classes.

Clinton Rempel tackles the uncomfortable phrase *teaching to the test*. His work for the CSU Chancellor's Office on "teaching via error analysis" identifies possible mathematical weaknesses reflected in the responses students give on the Elementary Level Mathematics (ELM) exam questions. He then offers students suggestions on avoiding these types of errors (see the CSU Math Success website at `http://www.csumathsuccess.org`).

The article by Angelo Segalla and Yong Hee Kim-Park on Bayes' Theorem offers some visuals that might evoke an interest in AP Statistics classes and perhaps in other parts of the high school mathematics curriculum.

Kimy Liu encourages teachers to use the number line to show students that *fractions are numbers on the number line*. Liu extends the usual idea of whole number arithmetic on the number line to fraction addition, subtraction, multiplication, and division .

Note that some articles are more anecdotal than others and do not use formal references. This is in line with the philosophy of this journal to have its contributors "write about what works for them in the classroom."

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Color Work to Enhance Proof-Writing in Geometry

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1. Introduction

Classroom teachers know that the use of color can be a powerful tool to keep track of and make sense of mathematical information. The introduction of colored chalk as well as overhead transparencies and colored markers into instruction in the second half of the 20th century was a revolution in teaching tools. The current uses of whiteboard and SmartBoard® with colored markers continues this tradition in instructional tools. The use of manipulatives and, to some extent diagrams, among learners has been researched and incorporated into the toolbox offered to current and future teachers (e.g., Friel & Markworth, 2009; Smith, Hillen, & Catania, 2007). However, the potential benefits of the uses of color in mathematics learning have not been systematically researched. While the research is new, the idea is not.

In 1847, Oliver Byrne published his reworking of Euclid's Elements, in which he used colored diagrams so extensively that the visual representations were inseparable from the proofs they were intended to support (see Figure 1). Published at a time when geometers' attention focused on non-Euclidean investigations, Byrne's work was not taken seriously, and was "regarded as a curiosity" (Cajori, 1928, p. 429). Byrne, however, did not intend his work for mere entertainment, but said the book enhanced pedagogy and encouraged retention of mathematical ideas by appealing to the visual. He suggested that by communicating Euclid's ideas through colorful renderings, instruction time could be used more efficiently and student retention increased (Byrne, 1847). One hundred and fifty years later, *why* this is the case is finally coming to be understood.

There are many ways that the use of color can reduce the difficulty of a problem situation without decreasing its cognitive complexity. Just as thinking about a phone number as three chunks of numbers allows us to remember a long string of

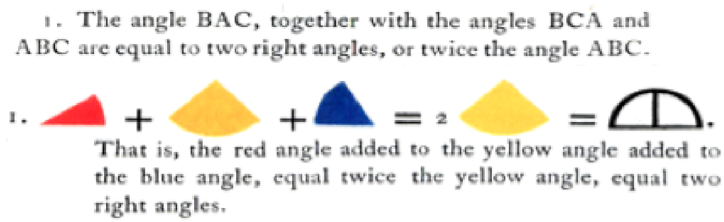


FIGURE 1. Example of a geometry statement first shown in words (translated from how Euclid expressed it), and then in colored shapes as Byrne (1847) represented it.

digits, the use of color can simplify the load on working memory and allow a learner to represent and strategize more efficiently (Paschler et al., 2007). Recent work in the learning sciences suggests that carefully selecting color in visual representations and combining information in a figure or symbolic expression can promote the integration of concepts. When presented with multiple sources of information (e.g., when a teacher relates parts of a mathematical equation to a graph or a student interprets a diagram), learners must direct their attention to each individual source, encode separate pieces of information, and then manage the stored information to make meaningful connections. Splitting attentional resources is cognitively demanding and may serve as an obstacle to learning. In fact, clinical research on the use of diagrams indicates that when individual sources of information are visually integrated, student learning is improved (e.g., Bobis, Sweller, & Cooper, 1993).

The work to date on color-coding for understanding symbolic grouping is further along than the work on color use in figures. In their research with pre-service elementary teachers, McGowan and Davis (2001) observed that students initially struggled to move from concrete manipulatives to algebraic expressions, and also struggled to see connections to binomial expansions. One student, however, conjectured that binomial expansions such as

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

could be re-interpreted through the use of two colors. The student represented a particular problem with a black and white color scheme and substituted black for a and white for b and restated the second-order equation using B for black and W for white, as $(B+W)^2 = B^2 + 2BW + W^2$. This idea resonated with the rest of the class and appeared in their subsequent work, indicating to the authors the algebraic symbols had "become genuinely symbolic – symbolizing something" (p. 441).

In working with secondary students to find a general rule for sequences of numbers, Waring (2008) used color to highlight relationships in pictorial representations of the sequences. Using red and blue to differentiate squares within each figure, students were able to correctly identify a sum of the squares of two numbers that related back to the figure number, n (see Figure 2).

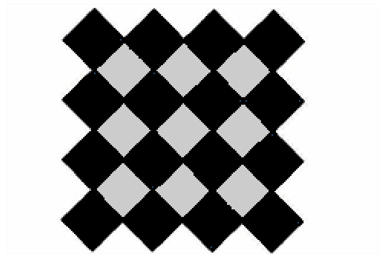


Figure 2. Red and blue (here, grey and black) coloring in the 4th figure of a sequence enabled students to see that the nth figure generalized to $n^2 + (n-1)^2$.

In research on student use of monochromatic figures, Gibson (1998) found that students use diagrams in several complementary ways:

- to understand information,
- to determine the truthfulness of a statement,
- to discover new ideas, and
- to verbalize their thinking.

Yestness (2012), in extending Gibson's work, noted that undergraduate students felt that their drawings were for personal use and not for proof or explanation. Nonetheless, when asked to explain a proof, students (and mathematicians) will draw one or more diagrams to support an explanation (e.g., Burton, 2004; Samkoff, Lai, & Weber, 2012). In fact, compact figural representations appear to be an intuitively powerful component of mathematical learning in the context of proofs and proving.

2. Nuances of Representation (models of) and Strategy (models for)

In investigating students' routes from informal mathematical activity to formal mathematical reasoning, Zandieh and Rasmussen (2010) explore models as "student-generated ways of organizing their activity with mental or physical tools" (p. 74). In particular, they specify a difference between models-of mathematical activity and models-for mathematical reasoning. It may be that students can use color in constructing diagrams of geometry proofs in this way – as a tool of representation, as well as a strategic tool for understanding.

In fact, the *Common Core State Standards* for mathematics, particularly the Mathematical Practices, highlight the kind of thinking supported by intentional color-coding. As noted in the Geometry strand, high school geometry students build upon elementary and middle school geometry content as they construct mental models for precise definitions and develop strategies for generating and validating proofs (CCSSO, 2010). In particular, the Mathematical Practices indicate that high school students are expected to: develop skill in abstract and quantitative reasoning, which includes practice creating representations of problems (Practice 2); construct arguments and evaluate others' arguments, which includes understanding and employing definitions, assumptions, and previous results for constructing arguments, while also communicating about and evaluating others' results (Practice 3), and appropriately and strategically using a variety of tools, such as paper and pencil (colored or standard), ruler, protractor, and dynamic geometry software (Practice 5).

3. Illustrating the Ideas: The Case of Charlotte

Here, we share what we are learning in our research on coloring and proofs. In particular, we focus attention on Charlotte Knight (a pseudonym), an undergraduate mathematics major preparing to be a secondary mathematics teacher, and her work while enrolled in a college course focused on modern geometry. Charlotte regularly employed coloring techniques in her proof-writing that were similar to the proofs offered by Byrne. Charlotte's representations enhanced her understanding in a way that may be of value to K-12 teachers and their students. We met with Charlotte for a task-based interview with two main components: first a review of one of the original colored proofs she submitted, in which she correctly proved that the diagonals of a parallelogram bisect each other, and then work on a proof covered in class, the Pointwise Characterization of Angle Bisectors Theorem:

> *Let A, B, and C be three non collinear points and let P be a point in the interior of $\angle BAC$. Then P lies on the angle bisector of $\angle BAC$ if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$.*

We had colored pens available on the table for her to use. Charlotte spent about 30 minutes of her 75-minute long interview describing how and why she used color to enhance her proofs. She also used color extensively in generating her proof of the Angle Bisectors Theorem (about 25 minutes). All four aspects of diagramming offered by Gibson (1998) and supported by Yestness (2012) were apparent in Charlotte's colored proofs.
In particular, Charlotte relied most on color in determining the truthfulness of statements and writing out ideas. She used color to confirm or refute ideas and to document the pathways she took. She also used it to reduce her cognitive load – she found it less mentally taxing to use color (rather than symbols or words). Including color served to help her sort and organize relationships, which she then used to write out her proofs. Charlotte used colors in two ways:

(1) as an organizational tool to connect her diagrams to the content of her proofs (i.e., as a tool of representation) and
(2) as a reasoning tool to understand the theorem (i.e., as a tool for understanding).

3.1. Color as a tool of representation.

In the proof she was asked to recount, where she proved the diagonals of a parallelogram bisect each other, Charlotte employed a 4-color scheme. She used these colors in a way in which the diagram was inseparable from the proof it was intended to accompany; she colored the angles to correspond to the underlined colors in her proof (see Figure 3).

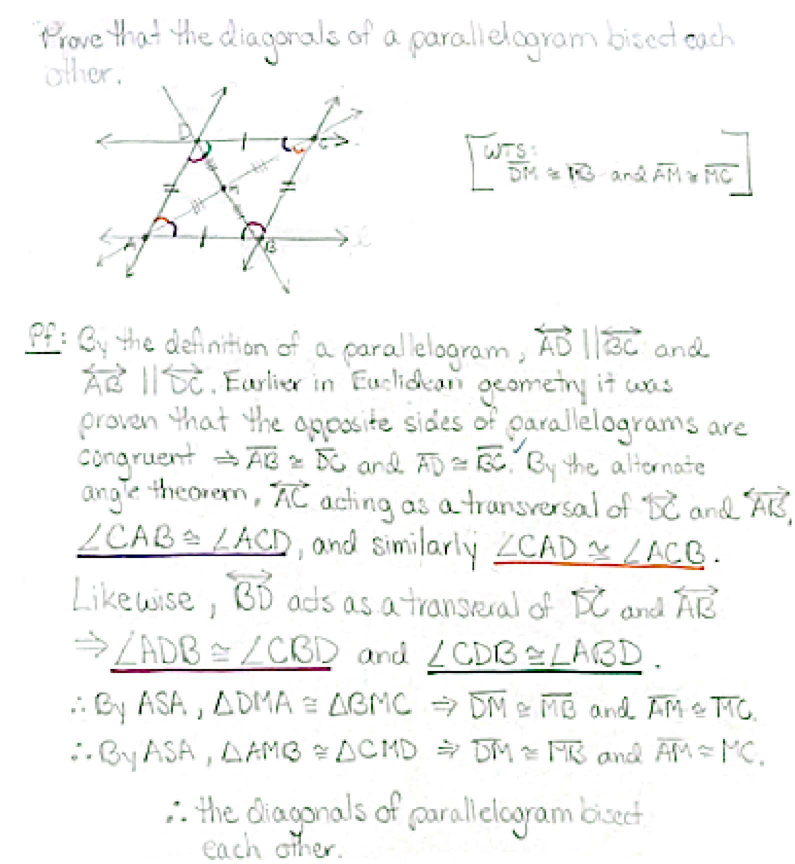


Figure 3. Charlotte's colored proof of the statement that the diagonals of a parallelogram bisect each other.

In describing this proof, she used the language "purple is congruent to purple," "orange is congruent to orange," "pink is congruent to pink," and "green is congruent to green." That is, the color replaced the alphabetical identifiers and this is how Charlotte navigated her proof:

> *I needed to look at, like, labeled the purple angles and then I underlined them for both so I knew purple was done . . . now which one is similar to the purple ones . . . to the orange ones and then I have pink and green left, well pink and then which one is similar to pink? Green. So that's how that, that's how that went.*

In recounting this proof, she spoke primarily of using colors to organize her ideas and understand the information required to write the proof.

3.2. Color as a tool for understanding. Charlotte regularly used color to indicate "direction" in a theorem. She did this when proving "if and only if" theorems, saying she was uncomfortable with these because she had difficulty keeping track of which "direction" she was proving, what information she could assume, and what she was trying to show. In her proof of the Pointwise Characterization of Angle Bisectors, Charlotte used a 2-color scheme. All information in the necessary "direction" was designated green and all information in the sufficient "direction" was designated blue. She then constructed and colored a diagram to reflect this information. As a result, to Charlotte, the statement of the theorem changed from "Then P lies on the angle bisector of $\angle BAC$ if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$" to "Then green if and only if blue" (see Figure 4). This served to help her reduce the cognitive load of attending to both implications in the "2 direction" theorem. It also aided her understanding of the information required to construct a proof:

> *I'm not as familiar with this picture ... so I needed to keep referencing back and forth here and so I needed to know ... it's kind of like a help to know where I'm going and it's, it's a reference.*

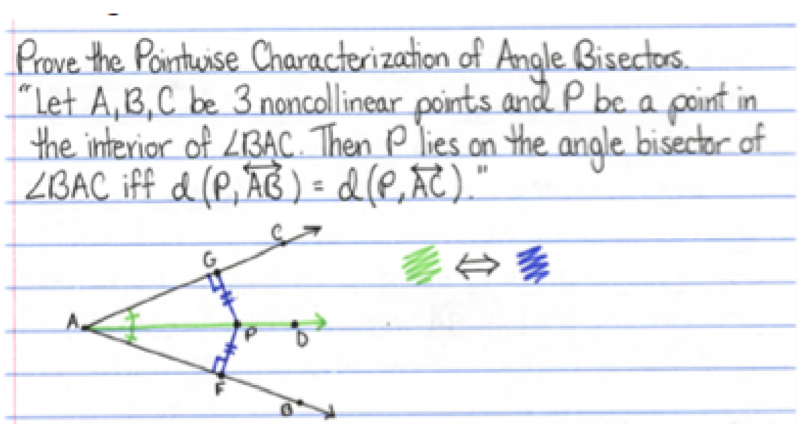


FIGURE 4. Charlotte's reworking of the statement of the Pointwise Characterization of Angle Bisectors theorem to "green if and only if blue" in a class homework assignment.

Charlotte said using the color helped her stay organized, understand the theorem, and stay on track with her proving goals:

> *This helps me remember which direction I'm going, 'cause all the green stuff is what I knew from the first half of the statement ... I put all of that in green.*

As she continued to construct the proof, Charlotte added a second layer of coloring – one in which she used color to understand and manage the mathematical content (see Figure 5).

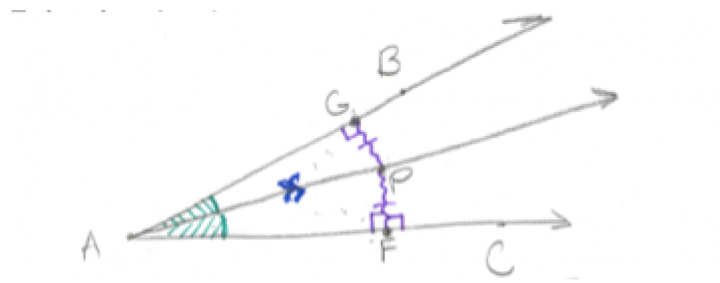


Figure 5. Charlotte's second drawing for the Pointwise Characterization of Angle Bisectors Theorem for "green is congruent to green if and only if purple is congruent to purple."

> *I don't have [segment] AG, I don't know anything about [segment] AG so there's no colors or label, there's no – nothing. I don't know anything about [segment] FA. What I do know is all in color, so it kind of helps me know, well, this is what I have to work with, because I don't want to go try to prove [segment] FA and [segment] FG, I don't have anything to work with to get there, so it helps that I have the purple angles here to say these are right . . . I don't think I used anything that wasn't related to color in some way. Like I'd never talked about just the segment FA, you know what I'm saying? I talked about segment AP, but I gave it a blue squiggle.*

4. Discussion

In advanced mathematics the prevailing wisdom is that pictures cannot prove. Students are discouraged from relying much on their visualizations when it comes to proofs and proving (Brown, 1997; Hanna, 2000). Charlotte agreed with this sentiment. She felt it was valuable to have a colored proof for her own sense-making. That is, a statement such as "If blue is congruent to blue, and purple is congruent to purple, then red is congruent to red" might be good for her notes. However, she asserted that without shared meaning, a proof such as this would not be a correct proof for "mixed company." Not only does Charlotte's view echo Byrne, it also illustrates something Martin Gardner said several years ago, "There is no more effective aid in understanding certain algebraic identities than a good diagram. One should, of course, know how to manipulate algebraic symbols to obtain proofs, but in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance" (Gardner, 1973).

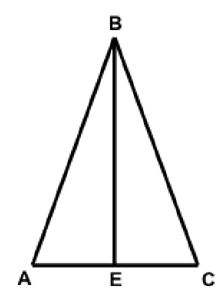
Mathematicians have the mathematical language mastery that allows them to navigate the formal symbolism of proofs. For students, use of the kinds of color-coding in visual representations discussed here may enhance understanding and may even serve as a viable proof-prepratation tool (Arcavi, 2003). While Byrne's (1847) assertion that using color-coding would allow students to see, at a glance, key parts of an argument generally has been affirmed by 20th century research on the mental "chunking" we do to manage complex information, Charlotte's work provides substantial support for this in the context of Geometry. Additionally, we noticed a growing number of students employing the use of color to support their diagrams in our advanced undergraduate mathematics classes – particularly those in which a majority of the students enrolled were seeking secondary mathematics teaching licensure.

As noted in the *Common Core State Standards*, some students use their experiences in high school geometry to develop Euclidean and non-Euclidean geometries as axiomatic systems. When students go on to college and prepare to become teachers, a collegiate geometry course is where students gain essential skills in visualization for "understanding the nature of axiomatic reasoning" and "facility with proof" (CBMS, 2000, p. 41). Yestness (2012) has observed that expanding pre-service teachers' experiences to include color-coding as a tool for their own learning, may "expand their pedagogical choices as teachers" (pp. 226-227).

5. Recommendations for Implementing Color in the Classroom

Although Charlotte was enrolled in an undergraduate modern college geometry course, high school geometry provides a similar context for teaching with color-coded proofs. The techniques might also be modified for use in the middle school classroom to prepare students for the transition to writing proofs in high school. Through our experiences using color to inform proof-writing in geometry courses at the undergraduate and secondary levels, we have identified five essential components for implementing color. We illustrate these recommendations by way of an example, generating a proof for the following:

Given:
Triangle ΔABC is an isosceles triangle with base $\overline{AC}$.
Segment $\overline{BE}$ is a median.
Prove:
Segment $\overline{BE}$ is an angle bisector.



5.1. Communication.

When incorporating color as a tool for understanding, explicitly identify and communicate the strategy. Multiple strategies may emerge during the proving process. Be specific about the use of color. For example, in the figure below, we employ color as a tool for understanding the prompt. All given information is colored green in the diagram and indicators for the statement that is to be proved are drawn blue, thereby changing the prompt to "If green is true, then blue must be true" (see Figure 6).

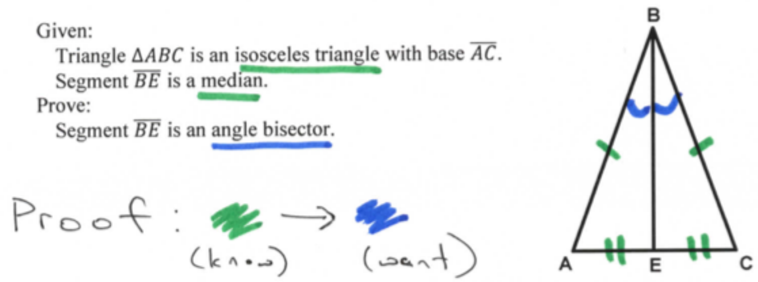


FIGURE 6. The statements in the prompt translate to "If green, then blue" in the diagram.

Continuing this process means the coloring scheme expands. It includes more colors as we use the scheme as a tool for representation (see Figure 7).

5.2. Purpose. Every color that you use should have a purpose – it captures some shared characteristics of the labeled parts. For example, in the diagram below, segment AE is congruent to segment EC. The purple double-ticks on AE and AC in the figure show congruency. We do not use purple again because there are no other segments necessarily congruent to these. We use green, red, pink, and blue in similar ways (see Figure 7). The monochromatic use of single or double ticks is enhanced with color as a tool for reasoning.

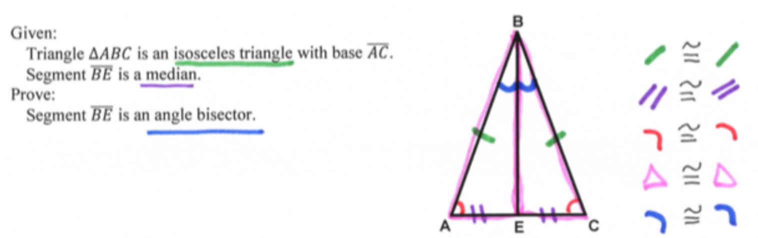


FIGURE 7. The diagram is colored to show "green is congruent to green" and "purple is congruent to purple." Therefore "red is congruent to red" and "pink is congruent to pink." Thus "blue is congruent to blue," completing the proof.

5.3. Incorporation. After completing a color-coded analysis of a figure or set of figures, the subsequent written proof should also incorporate the colors used in the figure analysis – either by writing, underlining, or highlighting in the appropriate colors. For example, in the two-column proof in the figure below the two congruent angles $\angle BAC$ and $\angle BCA$ are colored red in the diagrammatic proof. This is noted in the corresponding two-column proof by underlining the congruence statement in red. Other statements are similarly underlined in green, purple, pink, and blue (see Figure 8).

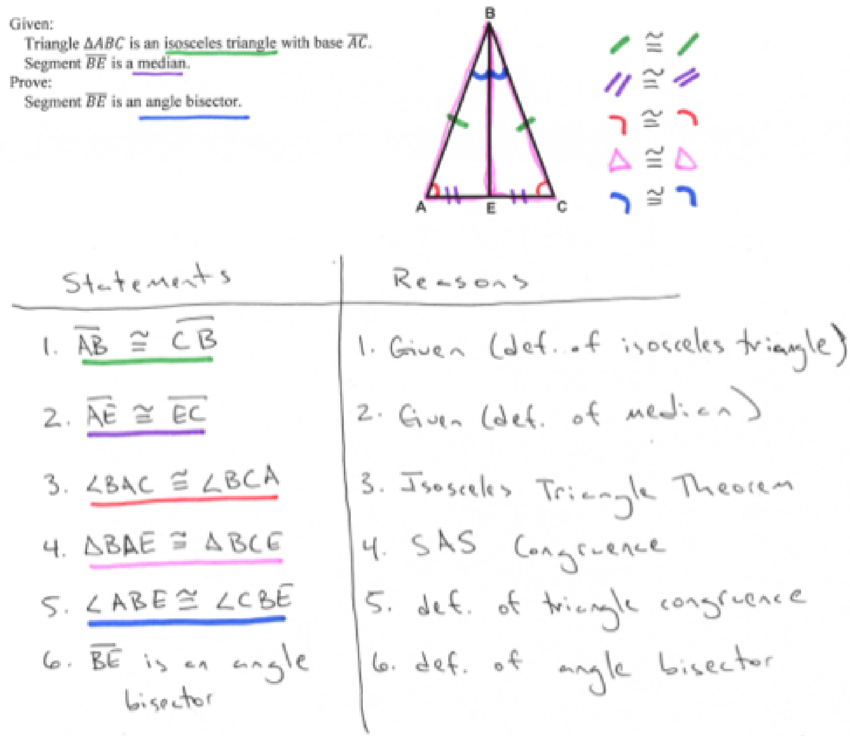


FIGURE 8. The "colored proof" has been translated into a colored two-column proof.

5.4. Consistency. When using color to teach geometry, be consistent with color use. Always make a legend to label the use of color, and consistently explain the property or characteristic captured or represented by the color in that color and in words. Students will be more inclined to use color in their learning process when it is a consistently modeled for understanding and representation. In Figure 6, in addition to coloring the "given" statements green and the "prove" statement blue, we included a note to accompany the diagram. In Figure 7, we included a colored legend to indicate congruences. This assisted us when we translated our colored proof into a traditional two-column proof in Figure 8.

5.5. Resources. Provide color tools in the classroom such as colored pencils or markers. This provides an equal opportunity for all students to participate by using color. Note that students may be more inclined to use color in their personal work when the tools are reliably available and their use expected in the classroom.

6. Conclusion

The utilization of color is not intended as a method for getting students' attention or as a means to make complicated drawings more attractive. Rather, through color-coding, relevant information in a proof is highlighted and significant relationships among components are foregrounded. Proof coloring is also beneficial for students to use as a learning tool on their own. A student can choose, strategically, how to color accompanying diagrams.

The strategies we have illustrated here are to color-code as a tool (1) of representation for facts and (2) for understanding of relationships. Such color-coding can assist students in packing and unpacking information and managing the complexity of proofs and proving. Furthermore, it may be that teachers can better assess how a student is approaching proof writing based on the color scheme utilized by the student. It provides teachers with a tool to help communicate with individual students and their different approaches to proof writing.

Acknowledgment

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Teaching To The Test, . . . Sort Of

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1. Introduction

To enroll in college level mathematics courses freshmen entering the California State University System must demonstrate "mathematical readiness," through one of the following criteria:

(1) SAT Math Reasoning Test: > 550,
(2) ACT Math: > 23,
(3) AP Math: > 3,
(4) Early Assessment Program (EAP): Exempt status, [1]
(5) College Course: C or better,[2]
(6) Entry Level Mathematics (ELM) Examination: > 50.

Simply put: "Readiness" is essential to success in collegiate mathematics and students who meet none of these qualifications are required to take essentially high school mathematics remedial courses at the university! Far from academically ideal and very expensive for taxpayers of California.

This article addresses a possible solution to improve the chances of those students who have none of the first five bypasses to qualify through (6), the ELM.

In-house research done at California State University, Long Beach (CSULB) shows that of the incoming freshmen taking the ELM, a population of students with an average B+ high school GPA, nearly half score below 50. [3] This is likely

[1] The California State University (CSU) requires high school students to take the English Placement Test (EPT) and the Entry Level Mathematics (ELM) exam prior to enrollment in the CSU unless they are exempt by means of scores earned on other appropriate tests such as the CSUs Early Assessment Program (EAP) tests in English and Mathematics, the SAT, ACT, or Advanced Placement (AP). See: CSU url on Admissions and Records.

[2] ibid

[3] Brown, C. "Analysis of ELM scores and GPA, California State University, Long Beach." 2005. McNair Scholarship Student Research Project. (unpublished.)

the case at many CSUs. As a result, at CSULB thousands of incoming students are channeled through courses equivalent to high school algebra: MAPB 1, MAPB 7, and MAPB 11 (MAth Pre Baccalaureate). This must be burdensome on University fiscal and personnel resources; as important, this is a significant time delay for students who pursue a baccalaureate, extending the time a four to five year academic program significantly.

Central to this article is the "Practice for ELM," sponsored by the Chancellor's Office of the California State University system, a program to be established for high school students to deal with this problem.

2. Background and Program Description

To assist high school students who will take the ELM qualify for college level courses, the California State University Chancellors Office contracted this author to develop an online practice test at for the CSU "Math Success" website where high school students could use to gain more mathematics sophistication, and yes, practice for passing ELM. See Figure 1.

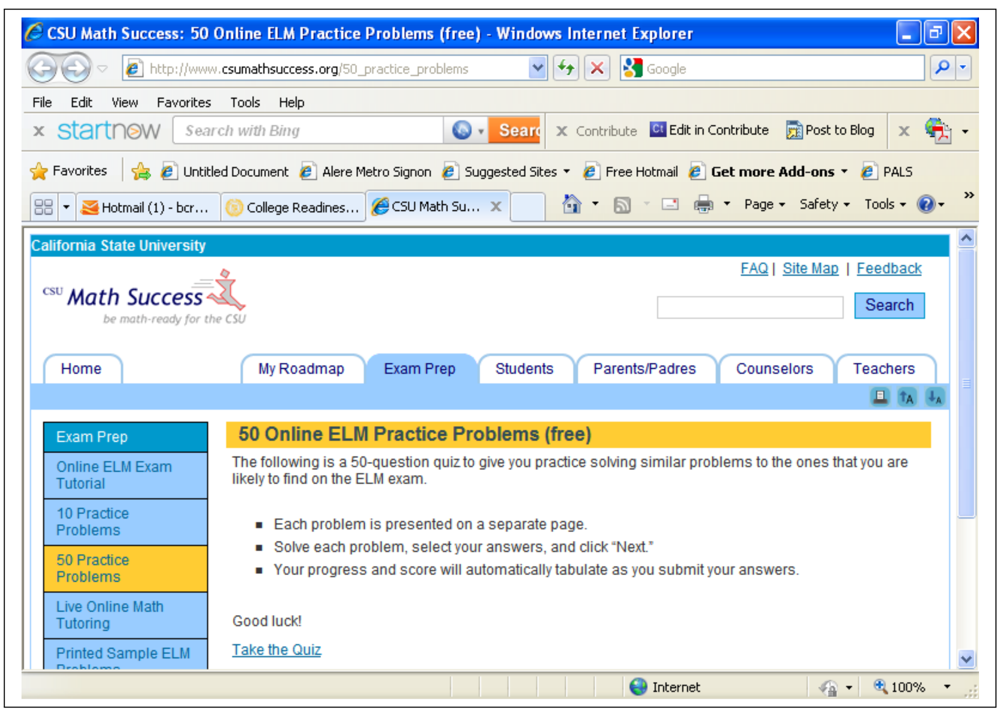


FIGURE 1. ELM Practice Exam Website

The "quizzes" on this site consist of released ELM test questions used in previous ELM tests. Figure 2 displays a typical question.

ENTRY LEVEL MATHEMATICS TEST

Time — 90 minutes

50 Questions

Directions: Solve each of the following problems and indicate your answer choice in the appropriate space on the answer sheet. You may use the blank space in this test book for scratchwork. However, mark all your answers on the separate answer sheet.

Notes: (1) Unless otherwise specified, the denominators of algebraic expressions appearing in this test are assumed to be nonzero.

(2) Figures that accompany problems are drawn as accurately as possible EXCEPT when it is stated that a figure is not drawn to scale.

1. Gina earns \$2,000 each month. If she spends \$200 on taxes, \$800 on rent, and \$400 on food each month, which of the following circle graphs best represents how Gina spends her earnings?

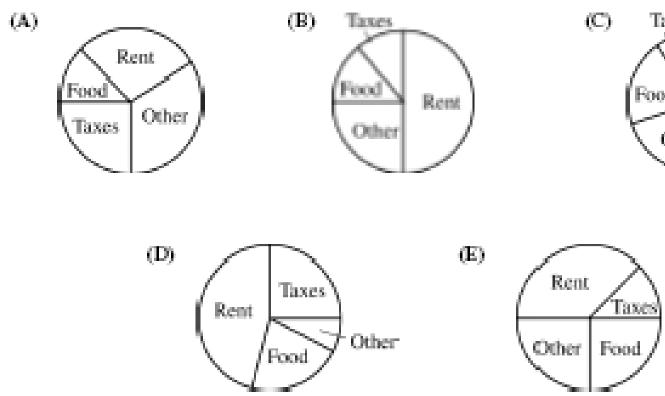


Figure 2. Typical ELM Practice Quiz Question

I found that this platform is amenable for “teaching to the test,” but in a positive fashion. I use it to teach the mathematics concepts high school students need to qualify for college level mathematics courses. The online practice familiarizes students with the various needed skills and concepts and, yes, also focus on the types of problems students will encounter on the ELM. Thus in a way we are “teaching to the test.” But the caveat is important: each wrong choice of answer on a given test item is accompanied by what we could label an “error analysis” that explains to the student *why* that particular choice is not correct. This is far from a face-to-face explanation, but worth the effort, I think.

Research on multiple choice tests indicates that wrong answers, “distracters,” are designed carefully by test writers to expose the subject’s conceptual and methodological errors, reflecting as much as possible the student’s competence in the material itself, and emphatically not, the student’s test taking skills. This is an unfortunate interdependence. Ideally, a test should expose the student’s weakness in the subject matter, not test taking skills.

I use the metaphor of "tip of an iceberg" to explore students' hidden misconceptions when looking for an explanation to wrong answers; the hidden mass of the iceberg. Suggestions in the answer part of the program described in the figures above are based the author's educated guess of what has gone wrong.

In descending order of frequency, I found the following to be the possible "below the surface" reasons for wrong answers:

(1) The student misunderstood or misinterpreted the problem (and the answer would have been correct for that interpretation).
(2) The student made an incorrect calculation or algebraic faux-pas.
(3) The student made a conceptual error.
(4) The student made a careless error.
(5) The student guessed at the answer.

There is a limited number of suggestions to the students for wrong choices "4. The student made a careless error" and to a lesser extent " 5. The student guessed at the answer". These have to do with study habits, and are a challenge to "repair" in any teaching-learning situation. Thus the first three reasons are the ones the program concentrated on.

"1.The student misunderstood or misinterpreted the problem (and the answer would have been correct for that interpretation)" is a balance between communication skills and mathematics, perhaps more the former than latter: EL comes to mind here.

Thus the "Teaching to the test" challenge here was advising students who had made a conceptual or skills error. The responses to wrong answers were guided mostly by these two reasons for an incorrect answer.

3. Teaching *Using* the Test

Our intent was to give meaningful feedback to students taking the practice test. For each wrong answer I concentrated on interpreting the student's thinking, retracing the flaw I thought I detected, and providing an appropriate hint or suggestion that might be helpful.

This is the meaning of the title "Teaching to the test . . . sort of." It is more a tutorial program than a teaching program. As such, the authors tried to "get inside the student's head" to discover where the mental process might need a road sign. No answers were given outright, just hints and suggestions to steer the student in the right direction; better *a* right direction.

There were fifty problems in the practice ELM, each with four distracters, so there were scores of possible incorrect solutions. The challenge to the authors was to identify "the left turn" or to use our metaphor, the bulk of the iceberg below the surface that represented the incorrect thinking, and use the hint do what was needed to make the correct choice on the next attempt. If the student chose another distracter, a different hint was given. Even if our hint did not exactly reflect the student's mental process, the student could compare and contrast his or her mental process with the process in the feedback.

4. Examples

Different questions and their distracters on the practice test revealed different shortcomings in a student's conceptual understanding and procedural skill.

For the sample question in Figure 2 we display below two types of responses. Figures 3 and 4 were the hints the authors planned. If the student picked distracter A, only the hint shown in the top half of Figure 3 would appear. Similar feedback is given for distracters B, D, and E in Figure 4. An approval statement was presented to support the correct answer C.

The reader will appreciate that this innocent looking problem is not so innocent; the student needs to know his or her geometry and use proportional reasoning.

Figures 5, 6, 7, and 8 offer a short explanation in the legend part of the figure.

Note: Each of the next six pages contains a single screen shot that takes up the entire page. This was done to make the screen shots more legible.

(A)

Note that the sum of the expenses for taxes and rent is $1000, which is *half* her monthly income. It is hard to say that (A) shows the sum of taxes and rent is half the circle graph. See table below

Gina's Monthly Budget			
Type of Expense	Amount	Approximate Percent	Degrees
Taxes	$200	10%	36
Rent	$800	40%	144
Food	$400	20%	72
Other	$600	30%	108
Total	$2,000	100%	360

(B)

Gina's monthly rent expenses, $800 plus taxes is half her salary, but (B) shows it to be more than half the circle graph. See table below.

Gina's Monthly Budget			
Type of Expense	Amount	Approximate Percent	Degrees
Taxes	$200	10%	36
Rent	$800	40%	144
Food	$400	20%	72
Other	$600	30%	108
Total	$2,000	100%	360

FIGURE 3. One possible explanation for choosing the wrong answer A or B to the problem in Figure 2.

(C)

Yes, Gina's monthly rent expenses, \$800, plus her \$200 for taxes adds \$1,000, half her salary, half the circle graph.

(D)

Gina's monthly rent expenses, \$800 plus taxes is half her salary, but (D) shows it to be more than half the circle graph. See table below.

Gina's Monthly Budget			
Type of Expense	Amount	Approximate Percent	Degrees
Taxes	\$200	10%	36
Rent	\$800	40%	144
Food	\$400	20%	72
Other	\$600	30%	108
Total	\$2,000	100%	360

(E)

Gina's monthly rent expenses, \$800 plus taxes is half her salary, and that is true in (E) but her "other" expense is \$600 and $\frac{600}{2000} = 30\%$ of the circle graph, not 25%. See table below.

Gina's Monthly Budget			
Type of Expense	Amount	Approximate Percent	Degrees
Taxes	\$200	10%	36
Rent	\$800	40%	144
Food	\$400	20%	72
Other	\$600	30%	108
Total	\$2,000	100%	360

FIGURE 4. One possible explanation for choosing the wrong answers D or E to the problem in Figure 2. Answer C is correct.

3. 0.215 − 16.215

(A) 16.43

You may have confused the rule for subtracting signed numbers by changing the sign of −16.215 and adding to get 0.215 − (−16.215)? This is not the case here, it is simply a subtraction problem like 3 − 10 = −7. Or, you might have thought that we cannot subtract a larger number from a smaller one. Indeed we can subtract any two real numbers.

(B) 16

You subtracted OK, but "in the wrong direction." Note that the second number is negative and it is larger in absolute value than the first number, so the difference should be negative.

(C) −14.065

−14.065 = 2.15 − 16.125. So your idea was correct, but you miscopied the 0.215 as 2.15. Try it again with 0.215.

(D) −16

Correct.

(E) −1643

Looks like you added: (−0.215)+(−16.215) = −16.43. This is a subtraction problem: +0.215−(+16.215). Try it again.

FIGURE 5. ELM Practice Exam Problem 3 on number sense.

10. Leni spends $\frac{1}{3}$ of her income on rent and $\frac{1}{4}$ of her income on car expenses. What fraction of her income is left for other expenses?

(A) $\frac{1}{2}$

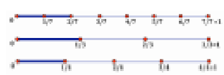
Keep in mind that 1 represents Leni's total salary, $\frac{1}{2}$ half her salary, and $\frac{1}{4}$ one-fourth her salary. Now $\frac{1}{3}+\frac{1}{4}$ is *about* $\frac{1}{2}$ but it is really $\frac{7}{12}$. You can think in terms of percent, if that is easier for you, then 1 is equivalent to 100%

(B) $\frac{1}{5}$

You started off OK by trying to add the fractions $\frac{1}{3}+\frac{1}{4}$, but you got $\frac{1}{5}$. Should be $\frac{7}{12}$, which then is subtracted from 1 to get the right answer. Please note: Perhaps you did not add the fractions correctly; we cannot simply add numerators and denominators. We need to first find a common (unit) denominator. Do not add fractions like this: $\frac{1}{3}+\frac{1}{4}=\frac{1+1}{4+3}$. This is bogus which makes $=\frac{2}{4+3}=\frac{1}{2+3}$ bogus. (We cannot cancel the 2's . . .why?) $=\frac{1}{5}$, which is the wrong answer. Should be $\frac{7}{12}$.

(C) $\frac{5}{7}$

You started off on the right foot by trying to add $\frac{1}{3}+\frac{1}{4}$. But you added the numerators and denominators, and then somehow got $\frac{1}{3}+\frac{1}{4}=\frac{1+1}{4+3}=\frac{2}{7}$. But this is not correct, nor is it the way to add fractions. Also, $\frac{1}{3}+\frac{1}{4}\neq\frac{7}{12}$. When adding fractions we cannot simply add numerators and denominators. The picture below shows you that $\frac{1}{3}+\frac{1}{4}$ cannot be $\frac{7}{12}(\approx\frac{1}{2})$. . .



(D) $\frac{5}{12}$

Correct.

(E) $\frac{11}{12}$

Looks like you did this: $\frac{1}{3}+\frac{1}{4}=\frac{1\times 1}{3\times 4}=\frac{1}{12}$. This is not the right way to add fractions. We cannot simply add numerators and denominators. We need to first find a common (unit) denominator. Do not add fractions like this: $\frac{1}{3}+\frac{1}{4}=\frac{1+1}{4+3}$. This is bogus which makes $=\frac{2}{4+3}=\frac{1}{2+3}$ bogus. (We cannot cancel the 2's . . .why?) $=\frac{1}{5}$, which is the wrong answer. Should be $\frac{7}{12}$.

FIGURE 6. ELM Practice Exam Problem 10; the student could make incorrect calculations and make conceptual errors in how fractions are added. Appropriate responses are given.

18. $\frac{(x-1)(3x+6)}{(3x-3)}$

(A) $x+2$

The idea is to simplify the numerator and denominator by factoring if we can to see if the numerator and denominator share common factors, which can then be divided out (or as is said more commonly, "cancelled.") Example: $\frac{14}{21} = \frac{2\times 7}{3\times 7} = \frac{2}{3} \times \frac{7}{7} = \frac{2}{3} \times 1 = \frac{2}{3}$. Here $\frac{(x-1)(3x+6)}{(3x-3)} = \frac{(x-1)(3)(x+2)}{3(x-1)} = \frac{3(x-1)(x+2)}{3(x-1)} = \frac{(x+2)}{1} \times \frac{3(x-1)}{3(x-1)} = \frac{x+2}{1} = x+2$ if $x \neq 1$.

(B) $x+6$

Looks like you did everything right except factoring $(3x+6)$; you got $3(x+6)$ instead of $3(x+2)$. Try it again

(C) $2(x-1)$

Looks like you wanted to "cancel" the $(3x+6)$ by $(3x-3)$ to get 2. (And ignore the minus.) If so, then your answer would be $2(x-1)$. BUT this is not right. Try factoring 3 from both $(3x+6)$ and $(3x-3)$.

(D) $-(2x-1)$

Looks like you wanted to "cancel" the $(3x+6)$ by $(3x-3)$ to get -2, but this is not right. Try factoring just 3 from both $(3x+6)$ and $(3x-3)$ to get $\frac{3(x+2)}{3(x-1)}$.

(E) $\frac{(x-1)(x+6)}{(x-3)}$

Looks like you wanted to "cancel" just the 3 from the 3x in $(3x+6)$ and the 3x in $(3x-3)$ to get $(x+6)$ and $(x-3)$, but this is not right. When you factor $\frac{(3x+6)}{(3x-3)}$ factor the 3 from both $(3x+6)$ AND $(3x-3)$ to get $\frac{3(x+2)}{3(x-1)}$.

FIGURE 7. ELM Practice Exam Problem 18 is an algebra problem.

The distracters tried to expose algebra errors. And again, for each answer, the authors provide a hint that may clarify students' thinking.

Age(years)	Percent of Students
22 or younger	44.7
23 - 25	25.2
26 - 35	20.1
36 or older	10.0

13. The table above shows the percent of students in different age groups at a university campus. The total number of students at the campus is 8,987. Approximately how many of the students at the campus are of age 25 years or younger?

(A) 2300

Looks like you used 25.2%. This represents *only* 23 to 25 year olds. We want *all* 25 year olds or younger; this includes the "22 or younger group," so we want 44.7% + 25.2% = 69.9% or about 70% of the 8987 students. Imagine all 8,987 students in the football stadium. The 22 years or younger sit in the stands on one side of the field, the 23, 24, and 25 years old sit in the stands opposite the first group, the 26 to 35 year olds in the stands behind one end zone, and the 35 years old and older on the opposite end zone. Which group is the largest? If 100% represent the 8987 students, then 44.7% + 25.2% = 69.9% or about 70%.

(B) 2700

Looks like you found the percent of students OLDER than 25: 20.1% + 10% = 30.1% and 2700 us about 30% of 8987. Now try to find the number of students YOUNGER than 25. You can rescue the situation by subtracting . . . or multiplying.

(C) 4,000

Looks like you computed the percent of students 22 years old or younger. 44.7% x 8987 = 4,017. But what about the 23 to 25 year olds? You need to add the two percentages and then multiply.

(D) 5,000

Looks like you found the percent of students OLDER than 22. You added 25.2% + 20.1% + 10% = 55.3% and 55.3% of 8987 is about 5,000. We want just the opposite: all students YOUNGER than 22 years old.

(E) 6,300

Good. 44.7% + 25.2% = 69.9% or about 70%, so 0.70 x 8987 is about 6300.

FIGURE 8. ELM Problem 13 involves interpretation.

The student could make interpretation errors. The problem is complex enough that the student could get confused in which process should be pursued. As with the other problems, each answer, wrong or right has a response that is hopefully helpful.

5. Conclusion

The research the authors did on the possible reasons for the wrong answers on the ELM practice test was interesting and hopefully helpful to students and insightful for teachers and student. Professional development for mathematics teachers could include analysis of distracters on multiple choice questions followed by plausible though processes leading to them. Another possible application of this approach is to help students develop their self-awareness. The feedback helps them think about what they did. The development of self-awareness is important in educational maturity, and the essence of mastering mathematics is correcting one's mistakes.

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Twenty Years Training Future Middle and High School Mathematics Teachers

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1. Introduction

There is an unspoken question in my head at the first day at beginning of each semester when I teach our entry-level credential course for hopeful future middle and high school mathematics teachers. As I scan the roomful of hopeful, smiling, expectant faces, and am about to begin the class, I think to myself: "So you think you want to be a high school mathematics teacher, uh?" My also unspoken answer is "OK, Im here to help you decide if this profession is for you, and if so, how to be the best possible teacher you can be."

I then heartily welcome the newbies to the credential program and to this first course in the program: "EDSS 300M: Introduction to Teaching High School Mathematics." [1]

This is my twentieth year in the mathematics department at California State University, Long Beach as the Single Subject Student Teaching Coordinator; teaching two, sometimes one, EDSS 300M courses, and placing, on average, about fifty student teachers in the local school districts schools each year along with assigning university supervisors for each student teacher. To keep my finger on the pulse of the program I also supervise, on average, eight student teachers.

My experience: four years at Torch Middle School, thirty years teaching mathematics at Gahr High School in Cerritos, CA, chair of the department and district mathematics coordinator for a 15 of those years.

Put this all together and I think I have come to recognize the traits necessary to succeed as a teacher in the middle and high school mathematics classroom.

[1] I shall refer to "high school mathematics" for short, but the program is geared to train middle and high school mathematics teachers. The single subject credential in mathematics allows its holder to teach mathematics in departmentalized classes grades K-12. Since grades K-6 are usually not deparmentalized, K-6 is usually the purview of multiple subject credentialed teachers.

2. First Days Of Class

After a general welcome and self-introduction of each student (I have each student memorize/internalize each of their classmates' names "from the get-go"), I begin with ten qualities and characteristics that a teacher should have to be successful (regardless of grade level and not necessarily order of importance):

(1) Have a caring personality.
(2) Memorize the student names quickly.
(3) Be extremely (and I mean extremely!) organized.
(4) Have a sound understanding of the fundamentals of mathematics.
(5) Be familiar with pedagogy in general, and that of teaching mathematics specifically.
(6) Break down large mathematics concepts and skills to incremental, connected pieces, in the manner of a spiraling curriculum.
(7) Develop meaningful lessons that:
  (a) are flexible (anticipatory),
  (b) involve group activities,
  (c) differentiate instruction, and
  (d) are guided by continuous formative evaluation like checks for understanding.
(8) Consistently enforce firm but fair classroom management.
(9) Be a good listener; kids have a lot to tell you–and teach you.
(10) Above all, be a team player with students and colleagues–even when the decisions go against your wishes.

We work on these and related teacher qualities the entire semester; class every meeting.

3. Example

Here is one of my course opening activities: "Think about the teachers you yourself have had and share each in turn with the class three positive characteristics." As simple as this is, it is a wonderfully enlightening experience for all, me included. We get nostalgic, amusing, surprising, and most of all, experiences that portend some important class goals.

They remember teachers who

(1) knew all the student names shortly into the semester,
(2) made it a point to ask students about their interests outside the math classroom, in other parts of their high school experience, say, the soccer team,
(3) had them explore the material as a special assignment, or in group activities,
(4) gave them projects that brought mathematics to life for them,

(5) used manipulatives to explore and enrich mathematics concepts,
(6) were attentive to students mathematical *and* individual needs,
(7) did not "just lecture" all the time.

And more. Too many to mention here, but the point is that this activity seemed to be central to their remembering successful teachers! Interestingly, these aspiring teachers rarely gave teacher mastery of subject matter as a criterion of (in their eyes) a successful teacher. I do not interpret this as suggesting that a firm grasp of the fundamentals of mathematics is not essential to be a successful teacher. Rather, it is positive affirmation that the teachers they remember were skilled enough in the subject matter that this was never an issue, a necessary but not sufficient quality of a successful teacher.

4. Student Teachers

The coursework (approximately one year) in the credential program in mathematics begins with my class: EDSS 300M, and continues with the California Department of Education (CDE) approved single subject credential program courses that include courses in Health Science, Psychology of the Adolescent, Education of Exceptional Individuals, Technology, to mention some, and culminates with EDSS 450M, our mathematics teaching methods course designed to prime our single subject candidates for the student teaching experience, the "trial by fire" experience of teaching and managing three classes with, of course, support from the master teacher and the university supervisor. Class management: the litmus test. While most candidates had earlier demonstrated a natural sense of what makes for good teaching, very few of the one hundred plus student teachers I supervised over the years had a sense of how much time and effort is required just to maintain good classroom management, before teaching the subject begins. "Successful classroom management does not just happen," I tell my candidates; "it takes a lot of thought, planning, the fortitude to stand firm but fair, and the decisiveness to follow through in an impartial manner." I continually review and remind class management techniques with my student teachers over and over again during the post-observation debriefing sessions. If I had to pinpoint one "bump in the road" to earning a single subject credential, class management would be it. Keeping students engaged in the mathematics material is another important class management skill! And this, of course, implies a solid understanding of the subject. [2]

5. Classroom Management, A Biggie

I am and have been concerned that there is a weakness in our program: our single subject credential candidates have to have at least 45 hours of observation in

[2]The reader may be aware that the "subject matter mastery" requirement of the single subject programs may be satisfied with either a CTC approved subject matter program, or the ability to pass the mathematics California Subject Examination for Teachers (CSET). This translates into a wide variation of subject matter mastery on the part of our candidates.

classes in the field during the EDSS 300M class, but with very little exposure in front of the class. The weakness is that they need to be more involved; observation is not enough. Just watching an experienced teacher, his or her seemingly easy classroom management style, has a tendency to lull our candidates into thinking that classroom management is easy. It is not, and too many of our candidates hit this brick wall when they student teach.

For example, grouping students seems a simple enough activity with the expected noisy thirty plus youngsters getting up from their seats, meandering around the room (yes, even after the teacher has made it clear how the activity is going to work), and finally, after the not unexpected stolen social greetings and fist-bumping hellos settling into the group seats, ready for the teachers instructions.

"It is not magic," I tell my credential candidates; "but you have work at it." I follow this advice with explicit directions and have the candidates act out the many techniques and examples of assigning students to groups.

I emphasize that classroom management techniques need not be dictatorial. Far from it, since this would ultimately be detrimental to young students. In my teacher training class we practice classroom management techniques that are friendly but effective, beginning with a warm greeting for each student at the door, ending with a farewell, punctuating the class period in between with noise-reduction-attention-getting techniques like "Clap your hands (once, twice, three times)."

We practice many other teaching techniques in my class, but formative assessment is another focal point in my teacher training class. Beginning with checks for understanding during each lesson, I believe this teaching-learning process technique is central to good teaching. Such assessment must be done often: every day, at the beginning, in the middle (especially in the middle), and at the end of each lesson. I stress the importance of beginning the lesson with a check that the necessary prerequisite skills have been learned, that students retained these necessary skillsyes, since yesterday! "Dont assume your students are going to remember" I remind our candidates. "Precede each new lesson with a small initial assessment; use checks for understanding of key points throughout the lesson, and end with an short exit assessment to make sure that the students learned the material."

In conclusion, there is, of course, a lot more than described here that I teach my future teachers in EDSS 300M. Yet, I always wish for more time in the semester to share with my students the many, many other, probably as important facts of successful teaching, but at in this course I do cover the more important ones.

Has it been worth it? Unquestionably yes! The icing on the cake is the flow of emails I get from some of our successful candidates. Here are but two recent ones I received:

> Hi Professor Lau, I just wanted to thank you again for being such an amazing University Supervisor during my teaching credential process at MCHS and CSULB! I really enjoyed working with and learning from you during this semester. I've grown so much and I honestly feel so much more prepared now than ever before to run my own classroom with efficiency and ease.
>
> Signed

> Professor Lau, I just accepted a position to teach math at XYZ High School. I want to thank you for helping me develop as a teacher this semester and being patient with me. I had valuable experience through the Cal State Long Beach Single Subject Credential Program and I am blessed to have learned from great educators. Again thank you for everything you have taught me.
>
> Signed

Yes, its worth it!

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Counting on Bayes' Theorem, or "Back to the Future"

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Dear Sir, I now send you an essay found among the papers of our deceased friend Mr. Bayes, and which, in my opinion has great merit and well deserves to be preserved.
A letter to John Bowlton, dated December 23, 1763 from Richard Price.

Abstract. The main thrust of this article is that Bayes' Theorem becomes plausible for high school AP Statistics classes since students are generally uneasy with conditional probability. A "Back to the Future" metaphor demonstrates the cleverness of the theorem, which never fails to surprise students. A simple example, followed by visuals that are quasi-proofs will hopefully enhance the conceptual basis of this important theorem. Finally, suggestions of more sociologically significant examples illustrate the theorem's statistical power.

1. Introduction

Bayes' (1701-1761) work "An Essay towards solving a Problem in the Doctrine of Chances" was published posthumously in 1763 by his friend Richard Price (see quotation). Also, Joseph-Louis Lagrange (1736-1813, born in Turin, Italy as Giuseppe Luigi Lagrancia), unaware of Bayes' publication, worked on the same topic and extended the theory in an essay of his own in 1774.

Bayes' Theorem, from a counting point of view, is a sensible approach for introducing probabilities in high school: first as fractions with decimals, then as fractions with a common denominator, and finally with problems whose probabilities are proper fractions with (usually) different denominators. This approach provides a bonus for high school students. It is a conceptual framework for understanding fractions through a systematic review of their properties (an often neglected skill in school mathematics).

We have found that Bayes' Theorem works nicely in high school AP Statistics classes. Our approach also reinforce students' facility with those rational numbers that are ordinarily called "proper fractions" (with different denominators), that is, fractions we get from working out probability problems, then as fractions with the same (common) denominator, then as decimals, and finally as percents. For example, the probability at least one "heads" on a flip of three honest coins is 7/8 or 0.875 or 87.5%.

Most introductory statistics textbooks have a section on Bayes' Theorem, usually following a discussion of conditional probability, and illustrate by examples of how *a priori* information about a compound event (an earlier event in a tandem of a sequence of two events) will change the calculation of the probability of a set of outcomes, often dramatically.

2. A Simple Experiment

Consider two bowls, Bowl A, and Bowl B. Bowl A contains 10 red marbles and 30 green marbles for a total of 40 marbles. Bowl B contains 20 red marbles and 20 green marbles also for a total of 40 marbles. Grand total: 80 marbles. All marbles are indistinguishable by touch. The list below summarizes the experiment.

STEP 1: Choose one of the two bowls at random, say by flipping a fair coin; the identity of the chosen bowl is not revealed.
STEP 2: Also randomly, choose a single marble from the given bowl.
STEP 3: The marble is green. This we know.
STEP 4: Find the probability that the green marble came from Bowl A. That is that Bowl A was the one chosen in STEP 1.

This problem, starting with the concluding event and recovering the antecedent event, forces us to "think backwards" to put it informally. Therefore, our subtitle "Back to the Future" as in the movie. Intuition would suggest that the marble most likely came from Bowl A since that bowl contains more green marbles than Bowl B. What percent of the time would our intuition be correct?

Symbolically, let A be the event of picking Bowl A, B be the event of picking Bowl B, G the event of picking a green marble, and R the event of picking a red marble.

In this example choose Bowl A or Bowl B with equal probability. So here $Pr(A) = \frac{1}{2} = 0.5 = 50\%$ and $Pr(B) = \frac{1}{2} = 0.5 = 50\%$. Let $Pr(G)$ and $Pr(R)$ be the probability of picking a green marble or red marble respectively (whether it be from Bowl A or B. We seek the probability that we picked Bowl A, given that the marble we have is green, G. Symbolically, we seek $Pr(A|G)$.

Now, reversing A and G (correcting the order), $Pr(G|A)$ represents the probability of picking a green marble given that we actually chose Bowl A. Since in our case there is an equal number of marbles in each bowl (a special condition to be sure), the combined number of marbles can be used to find the total probability of each color marble. This and other information are summarized in Table 1 where it is easily seen that $Pr(G|A) = \frac{30}{40} = \frac{3}{4} = 0.75 = 75\%$.

TABLE 1. Summary of probability using fractions, decimals, and percents.

	Bowl A	Bowl B	Total
Red marbles	$10/40 = 0.25 = 25\%$	$20/40 = 0.50 = 50\%$	$30/80 = 0.375 = 37.5\%$
Green marbles	$30/40 = 0.75 = 75\%$	$20/40 = 0.50 = 50\%$	$50/80 = 0.625 = 62.5\%$
Totals	$40/40 = 1.00 = 100\%$	$40/40 = 1.00 = 100\%$	$80/80 = 1.000 = 100\%$

Taking some liberty with probability theory, to be in the cell where "$30/40 = 0.75 = 75\%$" we must have chosen Bowl A, which occurs with $Pr(A) = 0.5$, and, since the ball is green, the probability that that green ball came from the entire collection of green balls is $(0.75)/0.625$. Putting all this together (again, informally) the probability of having chosen Bowl A given that the marble is green is $((0.75)(0.5))/0.625 = 0.6 = 60\%$.

3. Bayes' Theorem

Bayes' Theorem (Rule) tells us how to compute the *a posteriori* probabilities when we know information ahead of time, the antecedent. The probability of a hypothesis X in light of a piece of new evidence, Y, is

$$Pr(X|Y) = \frac{Pr(Y|X)Pr(X)}{Pr(Y)}$$

In our example:

$$Pr(A|G) = \frac{Pr(G|A)Pr(A)}{Pr(G)} = \frac{(0.75)(0.5)}{0.625} = 0.60 = 60\%$$

Why is this so? In Figure 1 we use Venn diagrams to describe our experiment visually. We take the interior of the rectangle to be the sample space S and its area to be equal to one. The next four diagrams illustrate the probability (area) that an event in sets X, Y, $X \cup Y$, and $X \cap Y$, respectively, occurs: $Pr(X)$, $Pr(Y)$, $Pr(X \cup Y)$, and $Pr(X \cup Y)$.

The last two diagrams are pertinent to our example. They demonstrate that the area of the intersection of X and Y, $(X \cap Y)$, does not change whether we are given that event X has occurred or event Y has occurred. This is key to proving Bayes' Theorem. This fact and some assumptions about conditional probability (which can be defined and proved) illustrate the answers to the questions we ask (and others we can ask) about our experiment. That is, the last two pictures demonstrate what happens if we are told, *a priori*, that "X has happened" so that all the points in the sample set S except for those in set X, including those in set Y are now excluded! Set X is now the new (total) sample space. Similarly for "Y has happened."

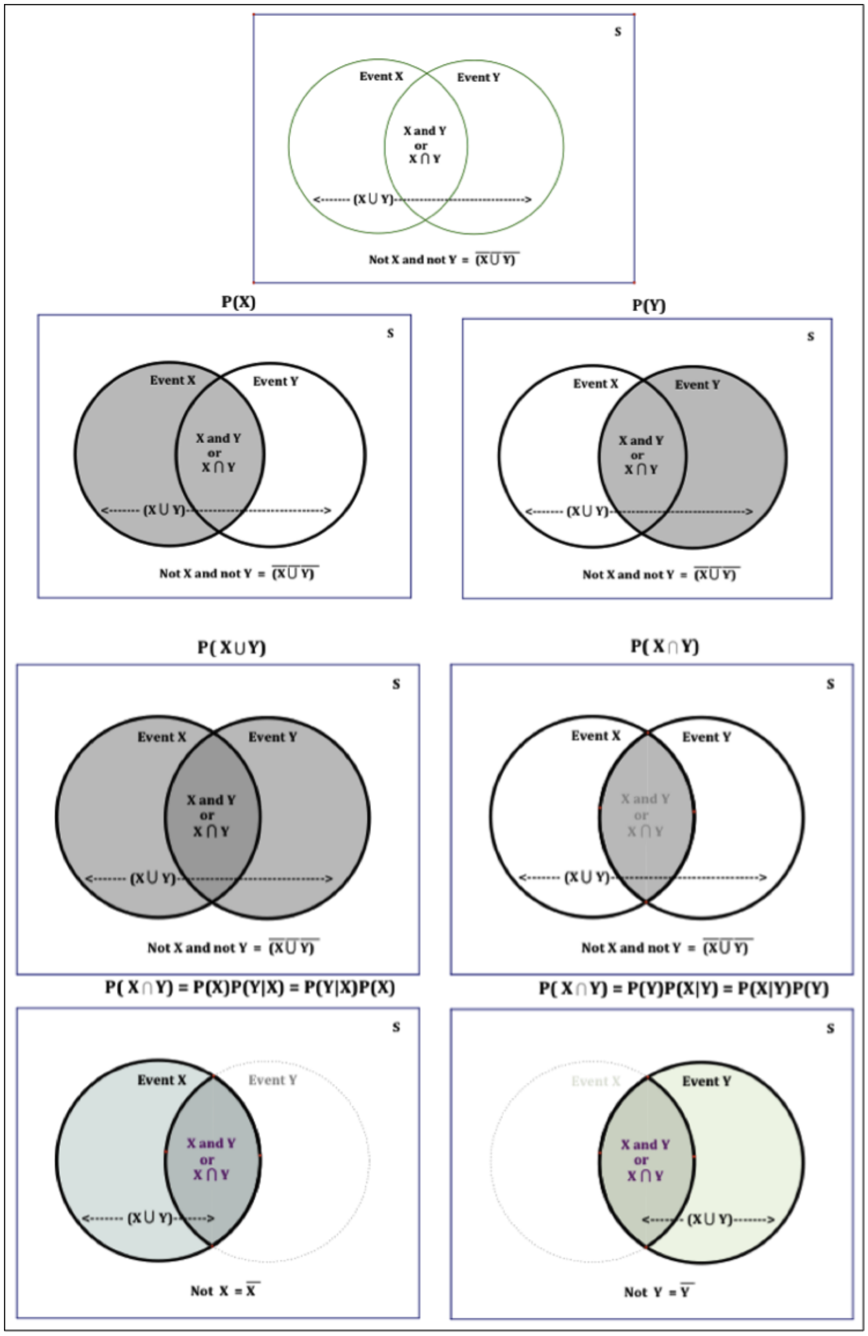

FIGURE 1. Venn diagrams for the sample space S, and sets X, Y, $X \cup Y$, and $X \cap Y$.

4. Proof of the Theorem

Now note in the diagrams that the area (probability) of $(X \cap Y)$ does not change if we are told that X has happened or Y has happened.

This hints that we can write the following if X happened

$$Pr(Y|X) = \frac{Pr(X \cap Y)}{Pr(X)}$$

and if Y happened,

$$Pr(X|Y) = \frac{Pr(Y \cap X)}{Pr(Y)}$$

But

$$Pr(X \cap Y) = Pr(Y \cap X)$$

So

$$Pr(Y|X)Pr(X) = Pr(Y \cap X) = Pr(X \cap Y) = Pr(X|Y)Pr(Y)$$

And

$$Pr(Y|X) = \frac{Pr(X|Y)Pr(Y)}{Pr(X)}$$

With the proper exchange of letters, the solution to the problem of the marbles is:

$$Pr(A|G) = \frac{Pr(G|A)Pr(A)}{Pr(G)} = \frac{(0.75)(0.5)}{0.625} = 0.60 = 60\%$$

And this backs up our intuition – but not overwhelmingly!

5. Trees

Yet another visually appealing method for simple probability problems, and simple Bayes' Theorem problems, that we found to work well in high school classes is illustrated in Figure 2 using a tree diagram. The answer to the question in our experiment (60%) is in the tree diagram, but needs to be interpreted. We leave this interpretation open for the reader with the hint that $37.5 + 25 = 62.5$.

Also note a tree diagram can elicit interesting questions to be posed by the teacher. For example, the sum of all the percents at the right of the tree must be $100\% = 1$, the probability of the entire sample space S.

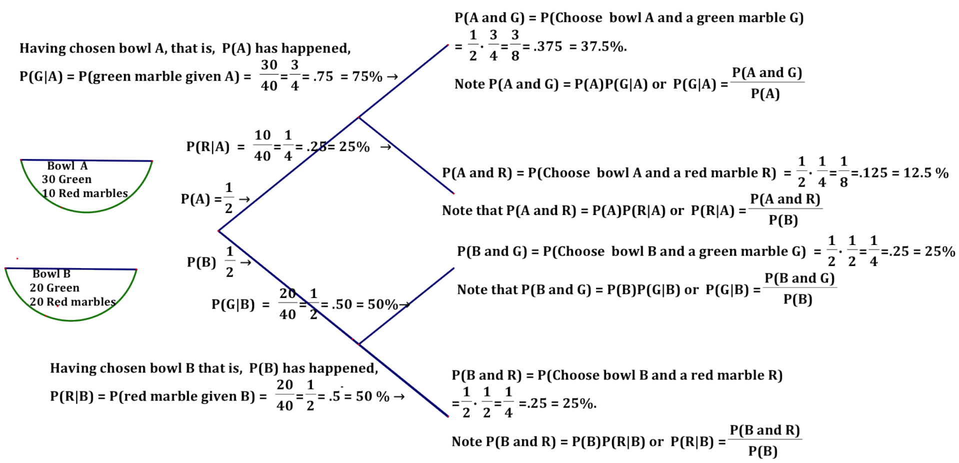

FIGURE 2. Tree probabilities of fractions, decimals, and percents.

6. Conclusion

The power of Bayes' Theorem can be appreciated even further in sociological examples that involve drug testing and profiling. Examples that challenge our intuition abound in the literature. Surprise can be the name of the game, so to speak, in statistics. Consider how on the evening of a presidential Election Day television networks project the winner with less than 1% of the votes counted!

Bayes' Theorem can play an important part in high school students' AP Statistics. Paired with the pedagogical emphasis on the different ways we can express rational numbers, the topic might also be used in other parts of the high school mathematics curriculum.

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Toward a Conceptual Understanding of Fractions Using The Number Line Model.

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There is a quote from the teaching of Zen to illustrate the journey of seeking deep understanding of concepts.

> *"Before I had studied Zen for 30 years, I saw mountains as mountains, waters as waters. When I arrived at a more intimate knowledge, I came to a place that I saw mountains not as mountains, and waters not as waters. But now I finally obtained the essence of the teaching and I could be at peace, for I saw mountains once again as mountains and rivers once again as rivers"* (Ch'uan Teng Lu).

1. Introduction

When teachers teach a new topic, when students try to learn something and "they cannot wrap their heads around it", they will go through the phase of "seeing mountains not as mountains." Even students comfortable with new concepts are often clumsy and awkward in conveying ideas to others. The tendency to follow, for example, an algorithm by rote memory and without understanding is real and tempting, for confronting misconceptions can be frustrating.

Teachers and students can demonstrate their perseverance by committing to this journey until the new information is fully integrated with prior knowledge, putting their minds once again at peace; when they "see mountains" once again "as mountains."

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2. The Number Line Model; The Unit

Students' fear of fractions is well documented (Ashcraft & Kirk, 2001) and it does not usually show up in the primary grades, when they learn the basic ideas and vocabulary of fractions such as "one-half, two-thirds, and three-quarters." Rather, this fear surfaces when they are taught to add and subtract fractions with different denominators, and the extensions to multiplication and division of fractions.

Fractions research shows that dividing the unit segment into fractional increments extends the number line concept already familiar to students from whole numbers to fractions, enabling learners to visualize the relative positions of fractions, including improper fractions and mixed numbers (Lamon, 2005; Wu, 2008). A learner begins by creating a number line using a paper strip, marking the positions of 0 and 1, to create a unit segment. Next, making a foray into fractions, the learner divides this unit interval, for example, into 10 equal parts, yielding the new unit of measurement of $\frac{1}{10}$. Extending the concept to non-unit fractions, the learner can next visualize two of those units, $\frac{1}{10}$ and another $\frac{1}{10}$, as two $\frac{1}{10}$s or "$2\frac{1}{10}$s" and eventually, a point on the number line called $\frac{2}{10}$. Similarly, the learner can divide the distance between 0 and 1 into 5 equal parts and mark off three $\frac{1}{5}$s or, as we soon see, $3 \times \frac{1}{5} = \frac{3}{5}$. That is, $\frac{1}{5}$ is the new unit of measurement and we have three such lengths on the number line.

Current literature shows that using the number line model facilitates the conceptual understanding of fractions for first time learners in the primary grades, as well as upper grade students, and teachers. According to Lamon, (2005), understanding "unitization" is the key to developing the conceptual understanding of fractions. The principle of unitization enables the learner to connect the prior knowledge of arithmetic of whole numbers to arithmetic of fractions.

How does this knowledge help the learner develop conceptual understanding of the mathematics of fractions? Here is a report on a professional development summer institute program we held last summer, at CSU Stanislaus.

Forty seven Grade 3-5 teachers participated in 40-hours of training using the "Sample Fraction Institute Model" developed by California Common Core State Standards in Mathematics Task Force (CaCCSS-M). One of the findings we learned from this summer institute was that teachers often have a restricted view of fractions, which begets erroneous rules in mathematical reasoning. For example, when we asked the participants in the summer institute "which fraction is closer to 1: 6/7 or 7/6? And why?", ten of 33 participants who gave the right answer also gave us a wrong justification such as "6/7 is closer to 1 because 7/6 is over 1" (answers obtained from the CCMP Pretest, 2012). It is possible that this reasoning stems from prior knowledge that fractions are always less than one, so they eliminated 7/6 as a viable option and chose 6/7 by default. This misunderstanding was observed repeatedly in the various class settings of pre-service teacher classes and in-service professional development. During the first day of the "Fraction Institute" when we asked the participants to provide an example of a fraction, the majority of responses were of the numbers less than 1.

Failing to include improper fractions as examples of fractions can lead to a narrow and restricted view of this important topic. To help the learners expand their views on the different types of fractions, we propose to use number line model to teach fractions.

3. Sample Lesson 1: Locate Fractions on the Number Line

The first lesson of learning fractions using the number line model is to help learners see a fraction, say, $\frac{1}{2}$, $\frac{3}{4}$, or $\frac{4}{3}$ as a point on the number line. At our summer Fraction Institute, participants first created a number line with whole numbers by first marking the unit segment on the number line and then using this unit of measurement to locate the numbers 2, 3 and 4. Next, they constructed another identical number line placed it parallel to the first and divided the line segment between 0 and 1 into 3 equal parts and used the new unit of $\frac{1}{3}$ to mark the rest of the number line. As participants marked the fractions $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, etc. on the number lines, they noticed that 1 and $\frac{3}{3}$ were the same point on the line, as were 2 and $\frac{6}{3}$, 3 and $\frac{9}{3}$. (See Figure 1.)

The use of the number lines helped students to expand their understanding of fractions to mixed number and improper fractions. Furthermore, use of the number line model transformed the concept of equivalent fractions from an abstract algorithm to concrete pictorial representations. Using the number line models, learners can rename fractions using the principle of unitization and then applying the schema of whole numbers to locate proper and improper fractions as well as the mixed numbers (See Figure 1). The following sample lesson shows how the participants applied the principle of unitization to locate fractions.

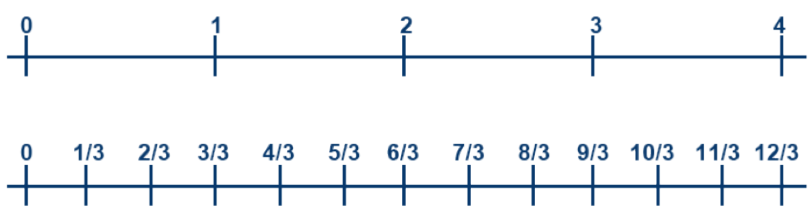

Figure 1: Locate the fractions on the number line.

Figure 2 shows a set of examples that challenged learners to investigate their present understanding of fractions. The task was to locate the fractions A, B, and C on the lines. At first, the majority of participants assumed that A, B, and C had the same value because they were on the same vertical line (same point) of three parallel and seemingly identical number lines. Upon further introspection, however, participants noticed the relative position of 0s and 1s on each number line. They noticed that two of fractions A and C were less than 1, and the fraction B was greater than 1. Thus, the number line model helped participants understand and visualize the value of a mixed number or improper fraction as a fraction whose value is greater than 1.

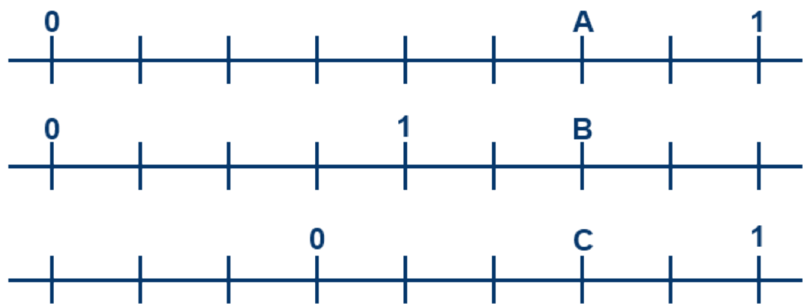

Figure 2: Find the fractions A, B, and C on the Number Lines

When asked how to determine the fractions of A, B, and C; here is a paraphrasing of what some said: To locate A, B, and C on the number lines, one must apply the knowledge of denominators and numerators. This helps to determine the "unit of measurement" in each number line. This is an application of unitization. For example, in the first line, there are eight equal parts between 0 and 1 where each part has a value of $\frac{1}{8}$. The point A is at the sixth mark after 0; therefore, it is at the position of $6 \times \frac{1}{8} = \frac{6}{8}$. In the second line, there are four equal parts between 0 and 1 where each part is $\frac{1}{4}$. The point B is at the sixth mark after 0, therefore, is at the position $6 \times \frac{1}{4} = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$. Although A and B appear to have the same numerator, they are of different values because the unlike denominators yielded in the different units of measurement in the respective number lines. Thus, to answer this question correctly, learners must have understanding on how to determine the unit of measurement on the number line.

The traditional pizza or pie analogy can be proportionally incorrect because the teachers were not able to easily partition the fractions in the proper units. They did not have a solid understanding of unitization and had a hard time judging whether or not answers they got were reasonable.

To measure anything requires a unit. Consider the non-example of putting a pile of dust on top of a second pile of dust. What do we get? Not two piles of dust, but a large pile of dust. This is because the concept of a pile is not a well-defined unit of measurement, unlike " one inch," "one apple," or "one a minute." The juxtaposition of examples and non-examples of "unit" gives the learner a better understanding of the principle. The learner will understand that the principles of unitization apply in the domain of whole numbers *and fractions*, but it does not apply in the domain of piles of dust, or other undefined units.

4. Sample Lesson 1: Locate Fractions on the Number Line

To add or subtract fractions, the learner must first transform the fractions into the same unit of measurement, after which the arithmetic of whole numbers can be applied. Just as we can rename one hour as 60 minutes, using unitization, the learner can rename a fraction. For example, 1 can be renamed as $\frac{3}{3}$ or $\frac{5}{5}$ or $\frac{100}{100}$. One-half can be renamed as $\frac{2}{4}$, $\frac{3}{6}$, $\frac{5}{10}$, or $\frac{10}{20}$. If learners understand the concept of unit and unitization, they will have conceptual understanding of

(1) the why and how of adding and subtracting fractions with like denominators,
(2) the why and how of converting fractions with unlike denominators to ones with like denominators, and
(3) the ability to articulate the rationale.

5. Sample Lesson 2: How to Add or Subtract Fractions with the Number Line Model: Find $\frac{4}{3} - \frac{1}{2}$ using the number line.

Figure 3a illustrates the same unit on the number line with three different sub-units: $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{6}$. With these common sub-units, the answer to the subtraction is easy: $\frac{4}{3} - \frac{1}{2} = \frac{8}{6} - \frac{3}{6} = \frac{5}{6}$
To check our answer we use the inverse of subtraction to go over the process on the number line in the reverse order. See Figure 3b without the hash marks. $\frac{4}{3} - \frac{1}{2} =?$ is the same as $\frac{4}{3} = \frac{1}{2} + ?$. The answer is displayed on the figure: $\frac{1}{2} + \frac{5}{6} = \frac{3}{6} + \frac{5}{6} = \frac{8}{6} = \frac{4}{3}$.

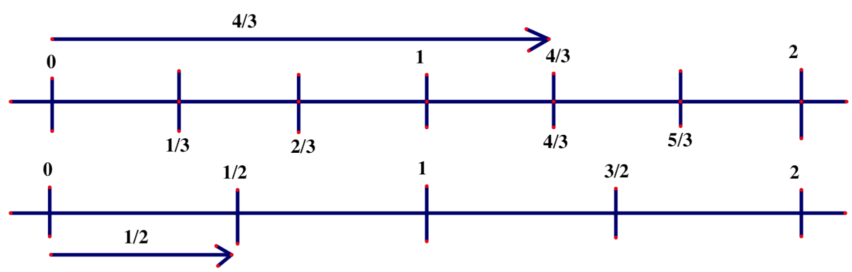

Figure 3a: Setting up the solution of the subtraction problem.

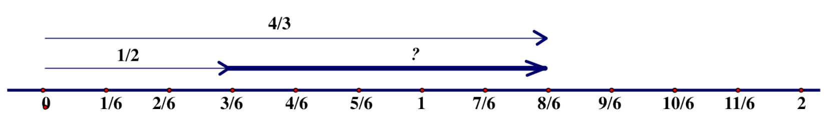

Figure 3b: Solution of the subtraction problem.

6. Sample Lesson 3: Multiplication and Division of Fractions

In addition to the adding or subtracting fractions with unlike denominators, the learner is also prone to feel challenged by multiplication and division of fractions conceptually. Multiplication of fractions, such as $\frac{2}{3} \times \frac{3}{4}$, can be represented with an area model and might be easier for the learner to understand. (See (Wu, 2008) for more details.)

Divisions of fractions, lacking a relevant schema to facilitate understanding seems the most foreign operation to teachers and students. In the absence of conceptual understanding, many teachers will use mnemonic of "KFC" ("Keep" the first one, "Flip" the second one, and then "Change" the division sign to multiplication) to teach students the procedure of solving division of fractions.

Unfortunately, without the conceptual understanding of the division of fractions this strategy could be over-simplified and confusing because they do not remember which fraction they should "flip." Furthermore there is not mathematical operation called "flip." More importantly, most teachers have difficulty explaining to the students *why* they should flip the second fraction in solving the division of fractions. This was part of the feedback from participants of our summer institute.

To overcome this learning barrier, the California Task Force suggests teachers use the number line model to help students make sense of the division of fraction problems.

6.1. Sample Lesson 3: How to Teach Division of Fractions: Find $2 \div \frac{2}{3}$ and $\frac{5}{8} \div \frac{1}{4}$ using the number line.

(1) Locate 2 and 2/3 on the number line. (See Figures 4a, 4b)
(2) Apply the arithmetic of whole numbers to the arithmetic of fractions: How many times will $\frac{2}{3}$ fit into 2?
(3) Use the inverse of division, which is multiplication or repeated addition, to check if the answer is correct.

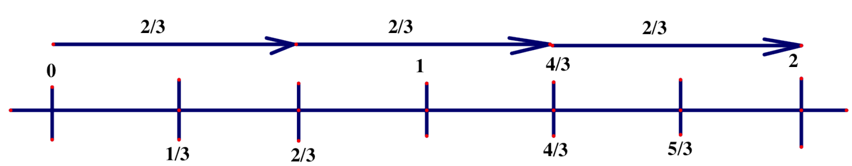

Figure 4a: How many $\frac{2}{3}$ can fit into 2?

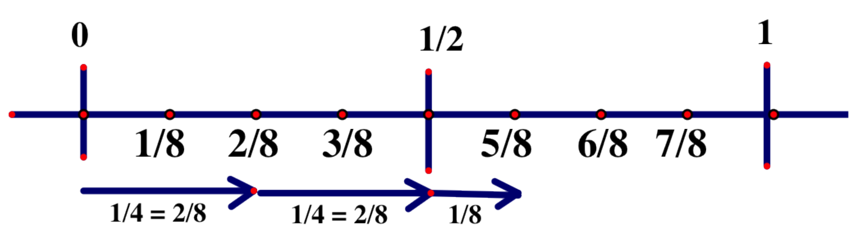

Figure 4b: How many $\frac{1}{4}$ can fit in $\frac{5}{8}$?

In both examples above, teachers can turn division of fraction, from an abstract expression, into a concrete example by using the number line.

Method: use two paper strips as the manipulatives. The learner can find how many $\frac{2}{3}$ of 1 unit can fit into 2, 1 units. The learner can also demonstrate the understanding that "the second number" in both problems is the new unit of measurement. To engage in academic discourse, the learner must know its proper name: *divisor.*

Moreover, the tangible notion of a fraction comparing the parts present to the parts that make a whole is typically a problem when students are told that a fraction is merely a division problem with the dividend called the numerator and the divisor called the denominator. Educators should compare the two notions, showing their common result. For example, the division notion of $15 \div 3$ is the number of groups of 3 that comprise 15. The fraction notion of $\frac{15}{3}$ is 15 units of $\frac{1}{3}$. Since 3 units of $\frac{1}{3}$ comprise 1, as the number line conveys, the task reduces to the division notion with the result of 5.

The ability to name the numbers correctly will help students in the process of conveying what they know or ask clarifying questions as needed. The learner can also use their prior knowledge and proficiency in the first language (division of whole numbers) to help them understanding a new concept and their second language (fractions).

7. Connection to the Common Core Standards for Mathematical Practice

For many learners, the mathematics of fractions is a foreign concept. The terminology and syntax used in the math discourse at times appear to be a second language. The understanding of unitization helps the learners to bridge the understanding of arithmetic of whole numbers (the schema of their first language) to arithmetic of fractions (similar schema presented in their second language). The uses of the number line help learners to think in pictures. When the learners have the right pictures in their heads, and words to accurately describe pictures in their heads, they gain proficiency in thinking, speaking, and reasoning with the mathematics of fractions.

In this process, the learner demonstrates the ability to "model with mathematics" (Common Core Standards for Mathematical Practice, Standard 4), "attend to precision" (Standard 6), "look for and make use of structure" (Standard 7), and "look for and express regularity in repeated reasoning" (Standard 8).

When a teacher engages students in mathematics practice as outlined in the common core standards, the teacher is guiding students to develop an in-depth conceptual understanding so as to "make sense of problems" (Standard 1) and "construct viable arguments and critique the reasoning of others" (Standard 3). When learners communicate with others about the process of finding equivalent fractions, they demonstrate their perseverance in problem solving and their ability to use language to convey their conceptual understanding.

Using the number line model, teachers can teach the specific abstract concepts including unitization, adding and subtracting fractions, multiplication or divisions of fractions with concrete manipulative (e.g. uses of paper strips) or pictorial representation. Helping students move flexibly between concrete examples, pictorial representations, symbols and mathematic formula of fractions can help students deepen their understanding and their ability to engage in academic discourse about fraction with others.

In the practice of discourse, teachers and students both deepen their understanding of fractions and values and uses of the number line models. In other words, they develop conceptual fluency of fraction sense, developing critical thinking skills while paving the road to success in algebra and geometry. At the end, the learners not only once again see the mountains as mountains; they can also describe the wonders they experienced while journeying through the mountains.

To effectively teach the concepts and uses of fractions using the number lines, teachers must integrate the four levels of depth of knowledge about fractions as defined by (Webb, 2005). In other words, teachers must first provide students a wide range of examples of fractions and show the minimal difference between fractions and non-fractions. Second, teachers should adopt the two-prong approach: (i) helping students develop the conceptual understanding of fractions by connecting it to their prior knowledge of whole numbers and real-life connections, and (ii) helping students to develop procedural fluency and conceptual fluency in making sense of fractions problems using the number line models.

To help the students with diverse learning needs, including students with limited English proficiency and math vocabulary, teachers need to model how to use the vocabulary correctly in the academic discourse. Teachers need to use the "think aloud" (van Someren, Barnard, & Sandberg, 1994) to make their cognitive and reasoning process overt to the learner. Ultimately, teachers need to understand that teaching fractions using the number lines is one of the vehicles to teach students to the Common Core Standards of Mathematical Practice.

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